Lagrangian Dynamics

Engineering Part IB
Paper 1
Lectures 9-16

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Contents

1 Course Details

1.1 Synopsis

Newtonian dynamics with constraints: Degrees of freedom, constraint forces, $F = ma$ in different coordinate systems, the energy method. Case study of the simple pendulum, sliding ladder and double pendulum.

Lagrangian dynamics: Generalized coordinates, the Lagrangian, generalized momentum, generalized force, Lagrangian equations of motion. Examples with one and multiple degrees of freedom. Relation between symmetry and conservation. Statement of time invariance and energy conservation.

1-D dynamics: 1-D equation of motion, first integral and energy conservation, motion in a potential energy well, stable and unstable equilibrium states, phase portraits, second integral of the equation of motion, case study of the simple pendulum.

Effective potential: use of angular momentum conservation to reduce 2-D problems to 1-D problems in an effective potential. Case study of radial potentials, including satellite motion and motion in a cone.

Two body problem: use of center-of-mass and separation coordinates to reduce the two body problem to one body motion in a radial potential.

Stability and normal modes: linearizing equations of motion, use of M and K matrices to find normal modes via generalized eigenvalue problem, expansion of the Lagrangian to identify M and K matrices, translational modes, orthogonal modes, normal coordinates, relation between stability/instability and real/imaginary mode frequencies.

Using a computer: different methods for numerical integration of equations of motion (Euler, Verlet, RK4) and notions of stability and order for integration methods. Case study of the double pendulum, poincare sections and chaos.

Time variation: understanding that Lagrangians can explicitly depend on time and how to incorporate external driving forces into Lagrangians, Case studies of an accelerating wedge, driven normal modes, parametric resonance.

1.2 Practicalities

This is a very new course: we taught Lagrangians in part I for the first time in 2019. Do a favor to future generations by sending typos, errors and general feedback to `jsb56`.

There is only one IB past tripos paper on this content (2P1 2019). However, to help with revision, we have also issued a sample paper (2P1 sample paper 2019) and a revision sheet with a few extra Lagrangian problems. These are available on the 2P1 moodle page. Additionally, much of this content was previously covered in 3C5. By the end of this course, you should be able to tackle Q4 (and, where there is one, Q5) on most 3C5 papers. Some particularly suitable 3C5 questions are listed on the examples sheets.

Many books and texts have been written on this subject: if you don't like my take, try one from someone else. A few good starting points are:

D.A. Wells, Lagrangian dynamics (Schaum's Ouline Series)

Goldstein, H. Classical Mechanics, Addison Wesley, 2nd edition 1980.

Introduction to Classical Mechanics (With Problems and Solutions) by David Morin.

Landau and Lifshitz, course for theoretical physics, vol 1: mechanics. This is a personal favorite — a short advanced book which covers everything in this course and much much more. Be warned though, short in length does not mean quick to read.

Finally, a beautiful set of notes on classical dynamics from David Tong at DAMPT:
http://www.damtp.cam.ac.uk/user/tong/dynamics/clas.pdf

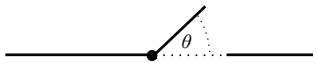
2 Generalized coordinates

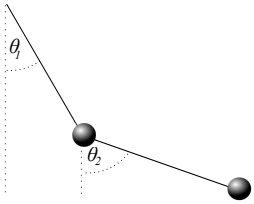
Dynamics is about predicting motion. For mechanisms, this means predicting how their configuration changes in time. Before we can do this, we need a set of quantities that describe the mechanism's configuration. The number of quantities you need is called the the mechanisms' number of degrees of freedom.

For example:

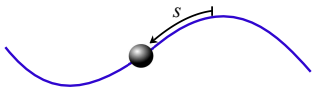
To specify the position of a particle in 3-D space, we need three Cartesian coordinates (x, y, z). The particle has three degrees of freedom, and its motion, is given by $x(t)$, $y(t)$ and $z(t)$.

A door has one degree of freedom, opening angle θ.





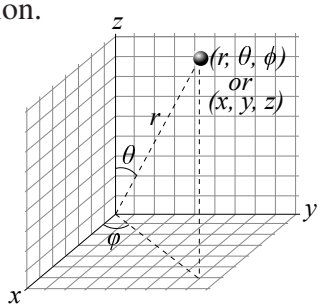
A double pendulum has two degrees of freedom, θ_1 and θ_2.



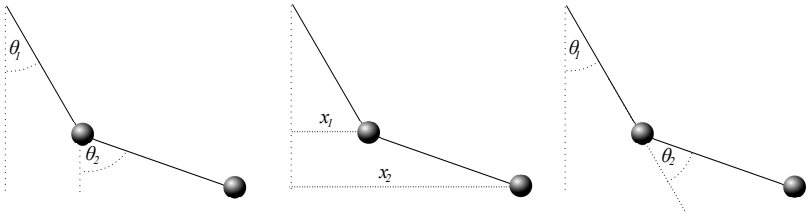
A bead that slides along a wire has one degree of freedom s.

A rigid body in three dimensions has six degrees of freedom: three for the location of its center of mass, and three angles (yaw, pitch and roll) to describe its orientation.

There are many different ways of describing the configuration of a mechanical system. For example, a particle in 3-D can be described with three Cartesian coordinates (x, y, z), three polar coordinates (r, θ, ϕ) three cylindrical polar coordinates (r, θ, z). However you always need to determine three quantities to pin the particle down.



Similarly, the configuration of a double pendulum could be specified in many ways, but you always need to specify two things.



We will call the set of n variables you are using to describe the configuration of an n degree of freedom mechanism it's *generalized coordinates*, and denote them $\{q_i\}$ = q_1, q_2....q_n, or sometimes as a vector $\mathbf{q} = (q_1, q_2...q_n)$. The use of q reminds us that we might be dealing with positions, or angles, or some combination of both.

3 Recap of Newtonian Dynamics

Dynamics is about predicting how things will move.

Newton provided a complete answer to this problem. All matter is a bunch or particles. Each particle has an *equation of motion* given by Newton's second law:

$$\mathbf{F} = m\mathbf{a},$$

which, written in 2-D Cartesian coordinates $x - y$ becomes:

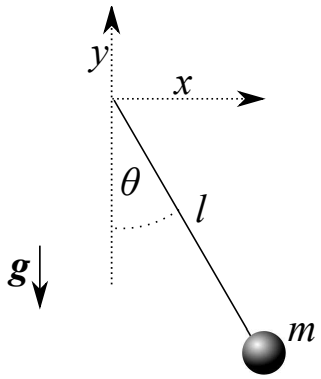
$$F_x = m\ddot{x}$$
$$F_y = m\ddot{y}.$$

Integrate these equations twice, $\ddot{x} \to \dot{x} \to x$, and you get the particle coordinates as a function of time. Problem solved.

Newton's scheme is easy if all the forces are known, and can be expressed simply in Cartesian coordinates. For example, if I release a stone of mass m, at rest, from height of $y = h$ then

$$F_y = -mg = m\ddot{y} \quad \Longrightarrow \quad \ddot{y} = -g \quad \Longrightarrow \quad \dot{y} = -gt \quad \Longrightarrow \quad y = h - \frac{1}{2}gt^2.$$

3.1 Simple pendulum



Life is harder when the particles are constrained in mechanisms. Now the most natural generalized coordinates will not be Cartesian positions of particles. For example, in a simple pendulum, the string constrains the mass to be distance l from the pivot:

$$x^2 + y^2 = l^2.$$

The string enforces this constraint by pulling on the mass with its tension, T, which is an example of a *constraint force*. If the mass tries to move further from the pivot, the string can always pull harder, but we don't *a priori* know what tension the string is producing, so if we are going to use $F = ma$, we must also solve for the tension.

Stupidly applying $F = ma$ to pendulum mass, in Cartesians, we have (noting that $\sin\theta = x/l$ and $\cos\theta = -y/l$)

$$-T\frac{x}{l} = m\ddot{x}$$
$$-mg - T\frac{y}{l} = m\ddot{y},$$

which, frankly, isn't very helpful. We have three equations, which we now need to solve to find the motion of the three unknowns, $x(t)$, $y(t)$ and $T(t)$. This is silly: the pendulum mechanism has a single degree of freedom described naturally by the angle θ, so all we really want to know is $\theta(t)$. To switch from Cartesian coordinates to θ, we can substitute $x = l\sin\theta$ and $y = -l\cos\theta$. This substitution directly satisfies the constraint equation and, with a bit of it of persistence (see

examples sheet!) one can then rearrange the equations of motion to get the famous pendulum equation and an expression for the tension:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \qquad\qquad T = mg\cos\theta + ml\dot{\theta}^2.$$

However, this approach gets very cumbersome with more complicated systems. We would prefer an approach that delivers the θ equation of motion directly. For the simple pendulum, several such approaches are on offer. One is to apply $F = ma$ in a polar r-θ coordinate system centered on the pivot. In A-level terms, this is equivalent to resolving parallel and perpendicular to the string. Looking in the data book for the form of the acceleration vector in polar coordinates, $F = ma$ becomes:

$$\begin{aligned} F_r &= m(\ddot{r} - r\dot{\theta}^2) \\ F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}). \end{aligned}$$

In polar coordinates the string constraint is simply $r = l$, which tells us that $\dot{r} = \ddot{r} = 0$, immediately reducing the $F = ma$ equations to the pair of equations we are looking for:

$$F_\theta = -mg\sin(\theta) = ml\ddot{\theta}, \qquad\qquad F_r = mg\cos(\theta) - T = -ml\dot{\theta}^2.$$

However, this approach relied on spotting a friendly coordinate system to simplify the constraint, which won't be possible for complicated mechanisms. Furthermore, even in this approach, half the work is going towards calculating T, whereas we only really care about the equation for θ.

A smarter approach is to use conservation of energy. The entire energy is the sum of the mass's kinetic energy, T (sorry) and its potential energy V:

$$E = T + V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta.$$

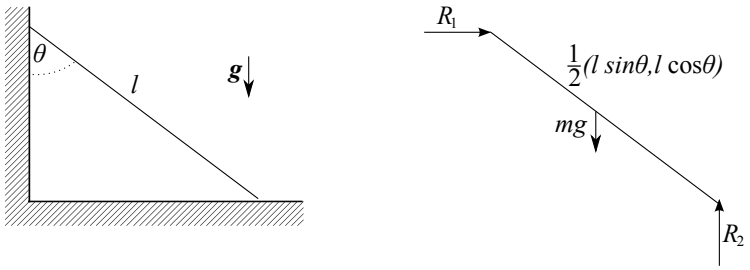
Mathematically, energy conservation means that $\dot{E} \equiv \frac{dE}{dt} = 0$, which gives us:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta\right) = 0 \\ \implies &ml^2\dot{\theta}\ddot{\theta} + mgl\sin\theta\dot{\theta} = 0 \\ \implies &\ddot{\theta} = -\frac{g}{l}\sin(\theta). \end{aligned}$$

This approach yields the θ equation directly, without tension making an appearance at all. It also bypasses difficulties of vectors and signs which plague $F = ma$. The approach works for any one degree of freedom mechanism that conserves energy, provided that you can do enough kinematics to work out the energy in terms of the natural generalized coordinate.

3.2 Ladder sliding down a wall

A second example is a ladder sliding down a wall without friction, as shown below.



Here the constraints are that the ends of the ladder stay in contact with the walls, and are implemented by normal reaction forces R_1 and R_2, as shown in the free body diagram.

The ladder is a rigid body, which we model as a uniform rod of total mass m. As a rigid body in 2-D, it is in general a three-degree-of-freedom system, with three coordinates, (x, y) for the location of its center of mass and θ for its orientation. From the theory of rigid bodies, we know that we can apply $F = ma$ directly to the center of mass giving two equations:

$$R_1 = m\ddot{x}$$
$$R_2 - mg = m\ddot{y},$$

and we have a third angular equation, $G = I\ddot{\theta}$ to describe the rotational dynamics, which we take around the center of mass to get:

$$R_2 \frac{l}{2}\sin\theta - R_1 \frac{l}{2}\cos\theta = \frac{1}{12}ml^2\ddot{\theta}.$$

To make progress, we again realize that the ladder really only has one degree of freedom, most naturally described by θ, so we substitute $(x, y) = \frac{l}{2}(\sin\theta, \cos\theta)$, which turns the two $F = ma$ equations into

$$R_1 = \frac{ml}{2}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)$$
$$R_2 - mg = \frac{ml}{2}\left(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta\right),$$

which are trivial equations for R_1 and R_2. Substituting these into the angular equation gives, after a good deal of algebra

$$\ddot{\theta} = \frac{3g\sin\theta}{2l},$$

which is the equation of motion we are looking for.

Alternativley, we could use the energy method. The kinetic energy of the ladder is the sum of a translational part and a rotational part

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m\frac{l^2}{4}\dot{\theta}^2 + \frac{1}{2}\frac{1}{12}ml^2\dot{\theta}^2 = \frac{1}{6}ml^2\dot{\theta}^2.$$

The potential energy is just the gravitational potential of the center of mass, so the total energy is

$$E = T + V = \frac{1}{6}ml^2\dot{\theta}^2 + mg\frac{l}{2}\cos\theta.$$

Using energy conservation, $\dot{E} = 0$ gives us directly

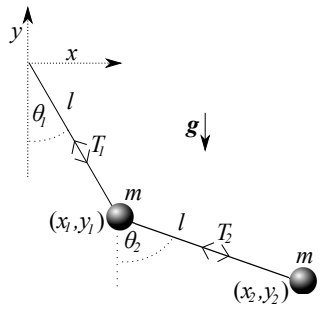
$$\dot{E} = \frac{1}{3}ml^2\dot{\theta}\ddot{\theta} - mg\frac{l}{2}\sin\theta\dot{\theta} = 0 \qquad \Longrightarrow \qquad \ddot{\theta} = \frac{3g}{2l}\sin\theta,$$

without any need to consider reaction forces, or any suppressed pages of tedious algebra.

3.3 Double pendulum

Finally consider a simple double pendulum consisting of two equal masses m connected by light strings of length l. The double pendulum has two degrees of freedom described by two natural generalized coordinates, θ_1 and θ_2. To find its motion, we need two equations of motion to calculate $\theta_1(t)$ and $\theta_2(t)$. Since $\dot{E} = 0$ can only deliver one equation, we are forced to go back to basics and apply Newton's laws directly. There is no helpful coordinate system, so we apply $F = ma$ to each mass in their Cartesian coordinates (x_1, y_1) and (x_2, y_2),

$$\begin{aligned} -T_1 \sin\theta_1 + T_2 \sin\theta_2 &= m\ddot{x_1} \\ T_1 \cos\theta_1 - T_2 \cos\theta_2 - mg &= m\ddot{y_1} \\ -T_2 \sin\theta_2 &= m\ddot{x_2} \\ T_2 \cos\theta_2 - mg &= m\ddot{y_2}, \end{aligned}$$



and remember we have the constraints

$$x_1^2 + y_1^2 = l^2 \qquad\qquad (x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2.$$

We need to solve these six equations to find $T_1(t), T_2(t), x_1(t), y_1(t), x_2(t),$ and $y_2(t)$. To satisfy the constraints, we write $x_1 = l\sin\theta_1$, $y_1 = -l\cos\theta_1$, $x_2 = l\sin\theta_1 + l\sin\theta_2$, $y_2 = -l\cos\theta_1 - l\cos\theta_2$. Substituting into the four equations of motion produces a complete mess:

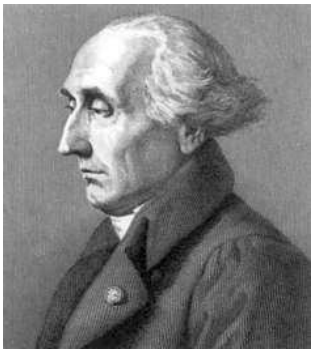
$$\begin{aligned} -T_1 \sin\theta_1 + T_2 \sin\theta_2 &= ml(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1) \\ T_1 \cos\theta_1 - T_2 \cos\theta_2 - mg &= ml(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1) \\ -T_2 \sin\theta_2 &= ml(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1 + \ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2) \\ T_2 \cos\theta_2 - mg &= ml(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1 + \ddot{\theta}_2 \sin\theta_2 + \dot{\theta}_2^2 \cos\theta_2). \end{aligned}$$

With a few pages of algebra, we can now solve these four equations for T_1, T_2, $\ddot{\theta}_1$ and $\ddot{\theta}_2$ to get:

$$\begin{aligned} T_1 &= -\frac{2m\left(2g\cos(\theta_1) + 2l\dot{\theta}_1^2 + l\dot{\theta}_2^2\cos(\theta_1 - \theta_2)\right)}{\cos(2(\theta_1 - \theta_2)) - 3} \\ T_2 &= \frac{-2m\cos(\theta_1 - \theta_2)\left(g\cos(\theta_1) + l\dot{\theta}_1^2\right) - 2lm\dot{\theta}_2^2}{\cos(2(\theta_1 - \theta_2)) - 3} \\ \ddot{\theta}_1 &= \frac{g(\sin(\theta_1 - 2\theta_2) + 3\sin(\theta_1)) + 2l\sin(\theta_1 - \theta_2)\left(\dot{\theta}_1^2\cos(\theta_1 - \theta_2) + \dot{\theta}_2^2\right)}{l(\cos(2(\theta_1 - \theta_2)) - 3)} \\ \ddot{\theta}_2 &= -\frac{2\sin(\theta_1 - \theta_2)\left(2g\cos(\theta_1) + 2l\dot{\theta}_1^2 + l\dot{\theta}_2^2\cos(\theta_1 - \theta_2)\right)}{l(\cos(2(\theta_1 - \theta_2)) - 3)} \end{aligned}$$

where the first two equations are expressions for T_1 and T_2, and the second two are the equations of motion for θ_1 and θ_2 that we actually want. Technically this all works, but it is very tedious, and things will only get worse for mechanisms with more than two degrees of freedom. We need a direct route to the equations of motion for θ_1 and θ_2 that doesn't go via constraints and Cartesian coordinates.

4 Lagrangian mechanics



Joseph-Louis (Giuseppe Luigi), comte de Lagrange

The energy method allows us to get the equations of motion for generalized coordinates directly, without needing constraint forces. We need an equivalent method, but for systems with more than one degree of freedom. In 1788, 100 years after Newton's principia, an Italian mathematician called Lagrange provided such an approach, which we now call *Lagrangian Mechanics*. Before we get to grips with it, it is helpful to start with two observations.

Firstly, Newton's second law of motion can be rewritten in terms of the momentum, $p_x = m\dot{x}$, as:

$$\frac{d}{dt}\left(p_x\right) = F_x.$$

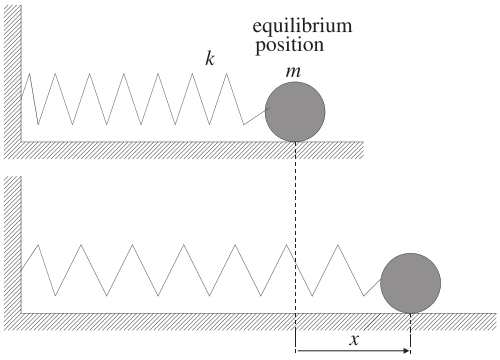
Secondly, a particle's momentum can be calculated from its kinetic energy T as

$$\frac{d}{d\dot{x}}\left(T\right) = \frac{d}{d\dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = m\dot{x} = p_x.$$

We will start with a general explanation of how to do Lagrangian mechanics, which (disclaimer) will appear quite bizarre, and simultaneously apply it to a very simple system, a mass m moving along the x axis and attached to a spring k

Consider a mechanical system with n generalized coordinates q_1, q_2, q_3....q_i...q_n. Remember, these generalized coordinates could be Cartesian coordinates, polar coordinates, angles, distances, or whatever else happens to be convenient.

For the mass, there is only one degree of freedom, and one generalized coordinate $q_1 = x$. For convenience we take $x = 0$ as the the spring's unstretched point.



We calculate the kinetic energy T and potential energy V, in terms of the q_i and $\dot{q}_i$.

For our mass, $T = \frac{1}{2}m\dot{x}^2$ and $V = \frac{1}{2}kx^2$.

Next we calculate the system's Lagrangian, $\mathcal{L}$, as a function of the q_i, $\dot{q}_i$ and in general time t, as

$$\mathcal{L}\left(q_1, q_2, q_3....\dot{q}_1, \dot{q}_2...., t\right) = T - V.$$

Notice the minus sign: the Lagrangian is *not* the total energy $E = T + V$.

For our mass:

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2.$$

For each generalized coordinate, q_i, we can calculate a *generalized momentum* p_i (also known as the momentum conjugate to q_i) and a *generalized force* F_i by taking derivatives of the Lagrangian:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad F_i = \frac{\partial \mathcal{L}}{\partial q_i}.$$

Notice these are partial derivatives. In this step, the Lagrangian must be regarded as a function of the variables $q_1....q_n$ and $\dot{q}_1...\dot{q}_n$, and all of these variables are held constant except the one the derivative is with respect to. For example:

$$\frac{\partial}{\partial q_1}\left(q_1^2 q_2 \dot{q}_1\right) = 2q_1 q_2 \dot{q}_1.$$

For our mass on a spring, these equations give the actual momentum and the actual force:

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad F_x = \frac{\partial \mathcal{L}}{\partial x} = -kx.$$

The equation of motion for each generalized coordinate is then simply:

$$\frac{d}{dt}\left(p_i\right) = F_i.$$

This result is obviously analogous to Newton's second law, but now written in a form that works for any convenient coordinate q_i. The time derivative, $\frac{d}{dt}$, is a full derivative not a partial derivative: it knows the coordinates and velocities are functions of time and catches everything. E.g.:

$$\frac{d}{dt}\left(q_1 \dot{q}_1\right) = \dot{q}_1^2 + q_1 \ddot{q}_1.$$

For our mass on a spring, this approach produces the expected $F = ma$ equation of motion:

$$\frac{d}{dt}\left(m\dot{x}\right) = -kx \qquad \Longrightarrow \qquad m\ddot{x} = -kx.$$

However, the amazing thing about Lagrange's approach, is that it works just as easily for highly constrained systems, non-inertial frames of reference and non-Cartesian coordinates, and therefore makes light work of hard mechanics problems. It delivers exactly one "$F = ma$" like second order equation of motion for each generalized coordinate.

In terms of the Lagrangian itself (rather than generalized momentum and force), the equation of motion for q_i can be written

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i},$$

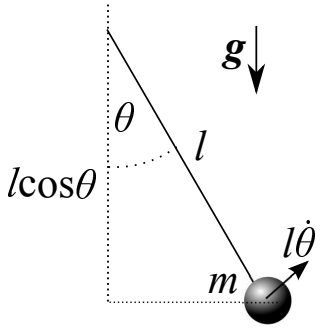
or, more commonly,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0.$$

The easiest way to get to grips with all this is via some examples.

4.1 Lagrangian examples with one degree of freedom

4.1.1 Simple pendulum



The pendulum is a one degree of freedom system, with coordinate θ. The velocity of the mass is $v_\theta = l\dot{\theta}$, and distance below the pivot is $l\cos(\theta)$. The Lagrangian is thus:

$$\mathcal{L}(\theta, \dot{\theta}) = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos(\theta).$$

The generalized momentum for θ works out to be mass's angular momentum around the pivot:

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta},$$

and the generalized force for θ works out as the gravitational torque around the pivot:

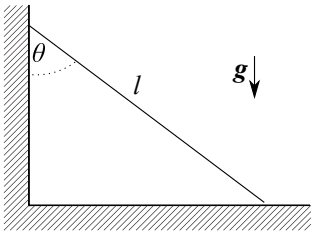
$$F_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta.$$

Putting these together, we get the correct equation for the pendulum:

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) = -mgl\sin\theta \qquad \Longrightarrow \qquad \ddot{\theta} = -\frac{g}{l}\sin\theta.$$

How easy was that! Like magic, we have ended up doing rotational dynamics, with angular momentum and torque, culminating not in $\dot{p} = F$ but its angular equivalent $\dot{J} = G$.

4.1.2 Sliding ladder



For the ladder sliding without friction, we previously worked out the kinetic energy as the sum of rotational and translational parts:

$$T = \frac{1}{8}ml^2\dot{\theta}^2 + \frac{1}{24}ml^2\dot{\theta}^2 = \frac{1}{6}ml^2\dot{\theta}^2,$$

and the potential energy as $V = mg(l/2)\cos(\theta)$. The Lagrangian is

$$\mathcal{L}(\theta, \dot{\theta}) = T - V = \frac{1}{6}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\cos(\theta).$$

The generalized momentum and force are

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{1}{3}ml^2\dot{\theta} \qquad F_\theta = \frac{1}{2}mgl\sin\theta,$$

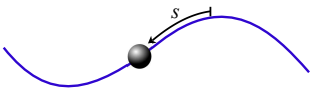
so the equation of motion, directly, is

$$\frac{1}{3}ml^2\ddot{\theta} = \frac{1}{2}mgl\sin\theta \qquad \Longrightarrow \qquad \ddot{\theta} = \frac{3g}{2l}\sin\theta.$$

If we were smart, we could have got this equation of motion directly by taking moments around $(x, y) = l(\sin\theta, \cos\theta)$, which is the point of intersection of the lines of action of the two reaction forces. However, Lagrangian mechanics has saved us from having to spot this.

4.1.3 Bead on a wire

A bead on a wire is described by the arc-length coordinate s, so its speed is $\dot{s}$ and its kinetic energy is $T = \frac{1}{2}m\dot{s}^2$. If the wire at s has height $h(s)$, the Lagrangian is



$$\mathcal{L}(s, \dot{s}) = T - V = \frac{1}{2}m\dot{s}^2 - mgh(s).$$

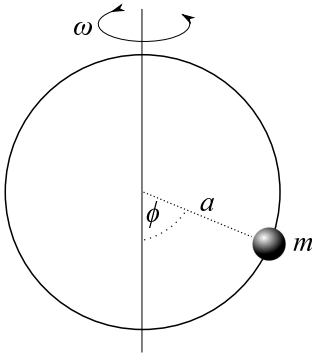
The generalized momentum and force are

$$p_s = \frac{\partial \mathcal{L}}{\partial \dot{s}} = m\dot{s} \qquad F_s = -mgh'(s).$$

We can recognize p_s as the momentum of the bead, and, if we recall that for a curve in intrinsic coordinates $h'(s) = \sin(\theta)$, the generalized force is the component of the weight force in the tangential direction. The equation of motion is

$$m\ddot{s} = -mgh'(s) \qquad \Longrightarrow \qquad \ddot{s} = -gh'(s).$$

4.1.4 Bead on a spinning hoop



As a final one-degree-of-freedom example, consider the motion of a bead on a spinning wire hoop, as shown on the right. The position of the bead on the hoop is described by a single coordinate ϕ, even though it is moving both along the hoop and with the hoop. The bead velocity can be broken into two orthogonal components: tangential to the hoop it is moving at $a\dot{\phi}$, and into the page it is moving in circles at $a\sin\phi\omega$. There is no potential energy, so the Lagrangian is simply

$$\mathcal{L}(\phi, \dot{\phi}) = \frac{1}{2}m\left(a^2\dot{\phi}^2 + a^2\sin^2\phi\omega^2\right).$$

The generalized momentum and force are

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ma^2\dot{\phi} \qquad F_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = ma^2\omega^2\sin\phi\cos\phi,$$

so the equation of motion for ϕ is

$$ma^2\ddot{\phi} = ma^2\omega^2\sin\phi\cos\phi \qquad \Longrightarrow \qquad \ddot{\phi} = \omega^2\sin\phi\cos\phi.$$

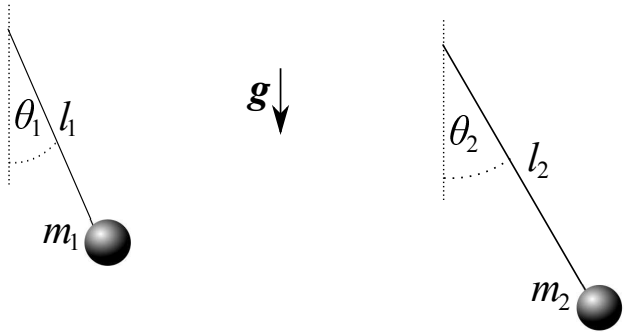
This is an amazing equation of motion to have got so easily. If you analyzed the problem in the frame rotating with the hoop, there would be a centrifugal force $ma\sin\phi\omega^2$ pointing horizontally outwards. Working in this frame, F_ϕ is the torque of this centrifugal force, p_ϕ is the angular momentum, and the equation of motion is $\dot{J} = G$: the Lagrangian approach has not only delivered an angular equation of motion, it has effectively done so in a rotating frame with the correct centrifugal torque!

4.2 Examples with multiple degrees of freedom

The real power of Lagrangian mechanics is in finding equations of motion for systems with multiple degrees of freedom. Unlike the energy method, Lagrangian mechanics delivers one equation of motion for each coordinate, which is enough to calculate the whole motion of the system.

4.2.1 Two simple pendulums

We first consider two entirely independent pendulums, with different length, mass, and angle.



The Lagrangian $\mathcal{L} = T - V$ is now the sum of the individual Lagrangians for each pendulum

$$\mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = T_1 + T_2 - V_1 - V_2 = (T_1 - V_1) + (T_2 - V_2) = \mathcal{L}_1(\theta_1, \dot{\theta}_1) + \mathcal{L}_2(\theta_2, \dot{\theta}_2),$$

where

$$\mathcal{L}_1(\theta_1, \dot{\theta}_1) = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + m_1 g l_1 \cos\theta_1 \qquad \mathcal{L}_2(\theta_2, \dot{\theta}_2) = \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 g l_2 \cos\theta_2.$$

If we now work out the generalized momentum and force for θ_1, we see that we get the answer we would expect from just using $\mathcal{L}_1$,

$$p_{\theta 1} = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \frac{\partial \mathcal{L}_1}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 \qquad F_{\theta 1} = \frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{\partial \mathcal{L}_1}{\partial \theta_1} = -m_1 g l_1 \sin\theta_1$$

and the right equation of motion:

$$\frac{d}{dt}\left(p_{\theta 1}\right) = m_1 l_1^2 \ddot{\theta}_1 = -m_1 g l_1 \sin\theta_1.$$

If we work out the equation of motion for θ_2 we will get the expected pendulum equation for θ_2. The general message here is that the Lagrangians for independent/uncoupled systems simply add together, and the motion for each sub-system will then be governed by its sub-Lagrangian. The Universe is one big system, and the Lagrangian for the universe is the sum of many independent sub-Lagrangians for its component parts. Indeed, since you asked (and obviously not for examination!) the Lagrangian for the Universe is actually the sum of three parts

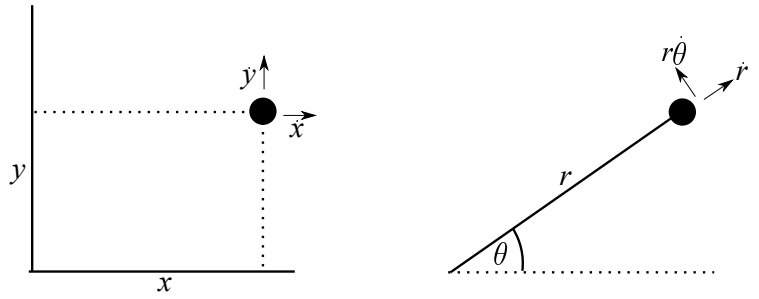
$$\mathcal{L} = \sqrt{g}\left(R - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \not{D} \bar{\psi}\right)$$

where the first term (and the $\sqrt{g}$) give gravity, the middle term is the other fundamental forces (electromagnetic and nuclear) and the last term gives "Leptons" like electrons and quarks.

More prosaically, this is also the first time we have produced equations of motion for two different coordinates, θ_1 and θ_2, from the same Lagrangian. We couldn't do that with the energy method, as we would only get one equation, $\dot{E} = 0$.

4.2.2 Particle in 2-D

Consider a particle sliding around a 2-D plane in a potential energy V. This particle has two degrees of freedom, which we might describe using the Cartesian (x, y) or polar (r, θ) coordinates.



In Cartesians, the velocity is simply $(\dot{x}, \dot{y})$ so the Lagrangian is

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - V(x, y).$$

The generalized momenta, generalized forces and equations of motion are

$$\begin{aligned} p_x &= \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} & F_x &= \frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial V}{\partial x} & m\ddot{x} &= -\frac{\partial V}{\partial x} \\ p_y &= \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y} & F_y &= \frac{\partial \mathcal{L}}{\partial y} = -\frac{\partial V}{\partial y} & m\ddot{y} &= -\frac{\partial V}{\partial y} \end{aligned}$$

which is exactly as you would get if you applied $F = ma$ in Cartesian coordinates.

However, in 2-D polar coordinated (r, θ) the velocity of the particle in the r direction is now $v_r = \dot{r}$, and in the theta direction it is $v_\theta = r\dot{\theta}$, so the Lagrangian for the particle is

$$\mathcal{L}(r, \theta, \dot{r}, \dot{\theta}) = \frac{1}{2}m\left(v_r^2 + v_\theta^2\right) - V(r, \theta) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta).$$

The generalized momenta and generalized forces are now

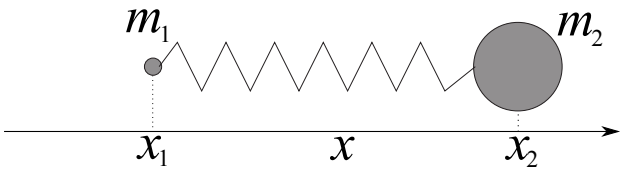
$$\begin{aligned} p_r &= \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} & F_r &= \frac{\partial \mathcal{L}}{\partial r} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2 \\ p_\theta &= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} & F_\theta &= \frac{\partial \mathcal{L}}{\partial \theta} = -\frac{\partial V}{\partial \theta}, \end{aligned}$$

we see that p_r is the particle's real momentum in the r direction, but the p_θ is its angular momentum about the origin. Putting these together into the equations of motion

$$\begin{aligned} \frac{d}{dt}(p_r) = m\ddot{r} = -\frac{\partial V}{\partial r} + mr\dot{\theta}^2 \qquad &\Longrightarrow \qquad m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r} \\ \frac{d}{dt}(p_\theta) = m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = -\frac{\partial V}{\partial \theta} \qquad &\Longrightarrow \qquad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -\frac{1}{r}\frac{\partial V}{\partial \theta}. \end{aligned}$$

We see here that we have got exactly $F = ma$, but now in polar coordinates. The complicated expressions for acceleration in the r direction ($\ddot{r} - r\dot{\theta}^2$) and the θ direction ($r\ddot{\theta} + 2\dot{r}\dot{\theta}$) have turned up in the Lagrangian equations automatically, with no need to look them up. Note also the importance of the full time derivative to correctly find the $\dot{r}\dot{\theta}$ Coriolis term.

4.2.3 Two masses connected by a spring



We next consider two masses, m_1 and m_2, each moving on the x axis, with coordinates x_1 and x_2, and connected by a spring of natural length l and constant k. This is a two degree of freedom system (x_1 and x_2), and the Lagrangian for the system is

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2\dot{x_2}^2 - \frac{1}{2}k(x_2 - x_1 - l)^2.$$

As the two masses are connected, this does not simply break into two separate sub-Lagrangians for each mass. The generalized momenta are the real x momenta of each mass:

$$p_{x1} = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1\dot{x}_1 \qquad p_{x2} = \frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2\dot{x}_2,$$

and the generalized forces are the real x forces:

$$F_{x1} = \frac{\partial \mathcal{L}}{\partial x_1} = k(x_2 - x_1 - l) \qquad F_{x2} = \frac{\partial \mathcal{L}}{\partial x_2} = -k(x_2 - x_1 - l).$$

Putting it all together, we have two equations of motion, which are the expected $F = ma$ equations for each mass:

$$m_1\ddot{x}_1 = k(x_2 - x_1 - l) \qquad m_2\ddot{x}_2 = -k(x_2 - x_1 - l).$$

This might not seem like much: we certainly could have got these equations with regular Newtonian mechanics. However, with Lagrangians we can easily analyze the system in more complicated coordinates. For example, we might instead take one coordinate as the center of mass $x_G = (m_1x_1 + m_2x_2)/(m_1 + m_2)$, and the other coordinate as the separation, $x_s = x_2 - x_1$. This is a different pair of coordinates to describe the same two-degree-of-freedom system.

In terms of these new coordinates, the actual positions of the masses are,

$$x_1 = x_G - \frac{m_2}{m_1 + m_2}x_s \qquad x_2 = x_G + \frac{m_1}{m_1 + m_2}x_s,$$

and, with a bit of algebra, the kinetic energy is now

$$\frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2\dot{x_2}^2 = \frac{1}{2}(m_1 + m_2)\dot{x_G}^2 + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}x_s^2.$$

The Lagrangian is thus:

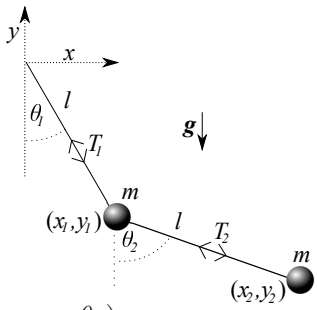
$$\mathcal{L}(x_G, x_s, \dot{x}_G, \dot{x}_s) = \frac{1}{2}(m_1 + m_2)\dot{x}_G^2 + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}x_s^2 - \frac{1}{2}k(x_s - l)^2.$$

In the new coordinates, the Lagrangian has turned into the sum of two separate parts, one for x_G and one for x_s, so these two coordinates will have uncoupled equations of motion. More precisely, the new generalized momenta, forces and equations of motion are:

$$\begin{aligned} p_{xG} &= (m_1 + m_2)\dot{x}_G & F_{xG} &= 0 & \Longrightarrow \quad & (m_1 + m_2)\ddot{x}_G = 0 \\ p_{xs} &= \frac{m_1m_2}{m_1 + m_2}\dot{x}_s & F_{xs} &= -k(x_s - l) & \Longrightarrow \quad & \frac{m_1m_2}{m_1 + m_2}\ddot{x}_s = -k(x_s - l). \end{aligned}$$

We see that p_{xG} is the total momentum of the two masses, and the x_G equation of motion corresponds to conservation of total momentum.

4.2.4 Double pendulum



Finally, we return to our original two-degree of freedom problem, the double simple pendulum. The Cartesian positions of the two masses are

$$(x_1, y_1) = l(\sin\theta_1, -\cos\theta_1) \qquad (x_2, y_2) = l(\sin\theta_1, -\cos\theta_1) + l(\sin\theta_2, -\cos\theta_2),$$

so kinetic energy for the double pendulum is

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2\right) \\ &= \frac{1}{2}m\left(l^2\dot{\theta}_1^2\right) + \frac{1}{2}m\left(l^2\dot{\theta}_1^2 + l^2\dot{\theta}_2^2 + 2l\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_1 + \sin\theta_1\sin\theta_2)\right) \\ &= \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right), \end{aligned}$$

and the potential energy is simply $V = mg(y_1 + y_2) = -mgl(2\cos\theta_1 + \cos\theta_2)$. Putting these together, the double pendulum Lagrangian is

$$\mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{1}{2}ml^2\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right) + mgl(2\cos\theta_1 + \cos\theta_2).$$

The generalized momenta and forces are then

$$p_{\theta 1} = \frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = ml^2(2\dot{\theta}_1 + \dot{\theta}_2\cos(\theta_2 - \theta_1)) \qquad F_{\theta 1} = \frac{\partial\mathcal{L}}{\partial\theta_1} = ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1) - 2mgl\sin\theta_1$$
$$p_{\theta 2} = \frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = ml^2(\dot{\theta}_2 + \dot{\theta}_1\cos(\theta_2 - \theta_1)) \qquad F_{\theta 2} = \frac{\partial\mathcal{L}}{\partial\theta_2} = -ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1) - mgl\sin\theta_2.$$

Putting all this together gives us the equations of motion for θ_1 and θ_2:

$$\frac{d}{dt}(p_{\theta 1}) = ml^2\left(2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_2 - \theta_1) - \dot{\theta}_2\sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)\right) = ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1) - 2mgl\sin\theta_1$$
$$\frac{d}{dt}(p_{\theta 2}) = ml^2\left(\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_2 - \theta_1) - \dot{\theta}_1\sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)\right) = -ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1) - mgl\sin\theta_2.$$

Neatening these equations up, we get

$$2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_2 - \theta_1) - \dot{\theta}_2^2\sin(\theta_2 - \theta_1) = -2\frac{g}{l}\sin\theta_1$$
$$\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_2 - \theta_1) + \dot{\theta}_1^2\sin(\theta_2 - \theta_1) = -\frac{g}{l}\sin\theta_2.$$

These equations are normally regarded as the final equations of motion, but to make contact with the previous Newtonian approach, we can solve these equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ to get the same final form.

Although this Lagrangian approach was still quite involved, it was far easier than applying $F = ma$ directly, and we haven't had to suppress any extra pages of algebra.

5 Conservation Laws

Emmy Noether

As we have just seen, complicated mechanisms with multiple degrees of freedom tend to produce complicated equations of motion. Often, we can't solve these equations, but we can try to understand what *type* of motion they produce. One powerful approach is to look for quantities, such as energy, momentum and angular momentum, which are constant during the motion. Lagrangian mechanics makes finding these quantities easy.

In general, the Lagrangian equation of motion is

$$\frac{d}{dt}\left(p_i\right) = F_i,$$

which, recalling how we calculate p_i and F_i, can be written in terms of the Lagrangian itself as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}.$$

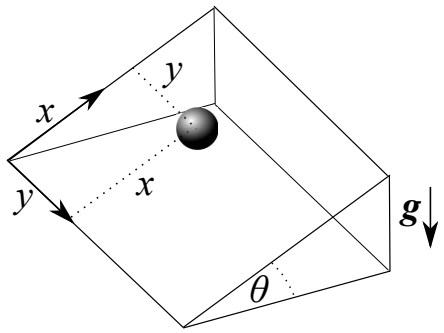
If it happens that the Lagrangian does not actually depend on q_i, then the generalized force is $F_i = \frac{\partial \mathcal{L}}{\partial q_i} = 0$ and the equation of motion now takes a very simple form:

$$\frac{d}{dt}\left(p_i\right) = 0,$$

which is simply saying that the generalized momentum p_i does not change during the motion: it is constant or *conserved*. This result is known as Noether's theorem.

We have seen examples where p_i is the real momentum, the angular momentum, and nothing easily identifiable, so this idea can prove the conservation of many different quantities depending on the system in question. Although the result seems simple — trivial almost — it is an amazingly useful, and has turned out to be one of the most profound insights in physics.

5.1 Example: Conservation of momentum



A particle, mass m, slides without friction on a two dimensional ramp that makes an angle θ with the horizontal. The particle has two degrees of freedom, described with Cartesian x and y coordinates in the inclined plane, as shown on the right. With these coordinates, the Lagrangian is

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mgx\sin\theta.$$

The generalized momenta are

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y},$$

which are just the x and y components of the real momentum. The generalized forces are

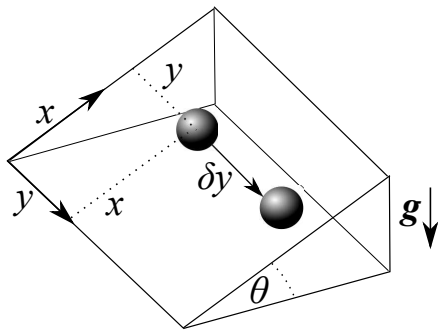
$$F_x = \frac{\partial \mathcal{L}}{\partial x} = -mg\sin\theta \qquad F_y = \frac{\partial \mathcal{L}}{\partial y} = 0,$$

which are indeed the actual total force on the particle in the x and y directions respectively. We note that $\mathcal{L}$ does not depend on y, so $F_y = 0$, and the equation of motion for y,

$$\frac{d}{dt}\left(p_y\right) = 0,$$

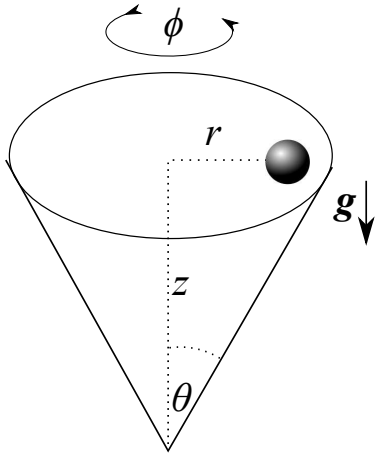
is a conservation law, telling us that $p_y = m\dot{y}$ is constant during the motion. The Lagrangian does depend on x, so the x equation of motion is not a conservation law:

$$\frac{d}{dt}\left(p_x\right) = -mg\sin\theta \qquad \Longrightarrow \qquad \ddot{x} = -g\sin\theta.$$



In Newtonian terms, these equations make perfect sense: the weight force has a component down the hill, so momentum down the hill is not constant, but there is no force in the y direction, so p_y is conserved. In Lagrangian terms, the Lagrangian not depending on y indicates that the Lagrangian does not change if you displace the particle a bit in the y direction, as shown on the right. This means that, if you release the particle in the same way from both starting points, the particle will conduct the same motion, just offset in y. The genius of Noether's theorem is that it links this invariance (or "symmetry") of the system to a conservation law in the motion: y momentum is conserved because the motion doesn't change if you start from a different y coordinate.

5.2 Example: Conservation of Angular Momentum



Consider a particle sliding without friction on the inside of a cone with opening angle 2θ, as shown on the right. The position of the particle is now described in polar coordinates $r - \phi - z$, though the particle only has two degrees of freedom since, on the cone, $r = z\tan\theta$. The Lagrangian for the particle is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - mgz,$$

which, if we eliminate z to use r and ϕ as our two coordinates, becomes

$$\mathcal{L}(r, \phi, \dot{r}, \dot{\phi}) = \frac{1}{2}m\left(\dot{r}^2\csc^2\theta + r^2\dot{\phi}^2\right) - mgr\cot\theta.$$

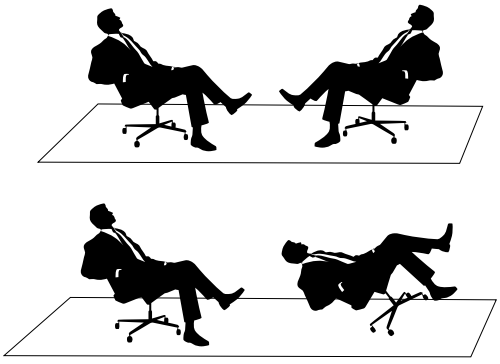
In this case, the momentum for the ϕ coordinate is

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi}$$

is the the z component of angular momentum. The Lagrangian does not depend on ϕ so $F_\phi = 0$ and the equation of motion for ϕ

$$\frac{d}{dt}\left(p_\phi\right) = 0$$

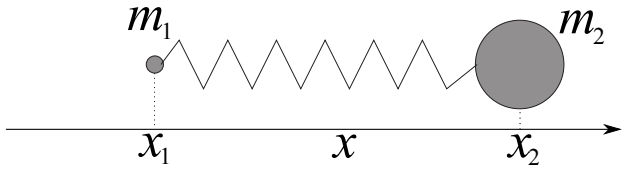
is the statement that z angular momentum is conserved. The Lagrangian does depend on r, so p_r will not be conserved.



An office chair is invariant under rotation about z (top), but not about x (bottom), so its motion will conserve $\mathbf{h}_z$ but not $\mathbf{h}_x$.

Just as linear momentum is conserved when the Lagrangian/system is invariant under a translational offset in space, angular momentum is conserved when the Lagrangian/system is invariant under an offset in angle, i.e. a rotation. In this case, if we start the particle in the same way from two locations with different values of ϕ, the two motions will be identical but offset in ϕ, and this invariance leads to conservation of angular momentum. Once we learn to think in this way, it gets much easier to spot when we expect certain components of angular momentum to be conserved. For example, imagine a spinning office chair. If you rotate the chair 90 degrees around the z axis, nothing really changes — the system is invariant — so its motion will preserve angular momentum about the z axis. A spinning chair continues to spin. On the other hand if you rotate 90 degrees around the x axis, you will fall out of the chair. This is far from invariant, so the system will not conserve angular momentum around the x axis.

5.3 Example: Conservation of total momentum



Consider, again, the system of two masses and a spring from section 4.2.3. In this case, we do not expect either mass to conserve its individual momentum, as the two masses are pushing and pulling on each other via the spring. However, as there is no external force on the system, we do expect it to conserve its total momentum. If we take our two coordinates as x_1 and x_2, the positions of the masses, the Lagrangian is again

$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2\dot{x_2}^2 - \frac{1}{2}k(x_2 - x_1 - l)^2.$$

As we saw previously, with this coordinate choice, the generalized momenta are the real x momenta of each mass, ($p_{x1} = \frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1\dot{x}_1$) and, since the Lagrangian does depend on both x_1 and x_2, neither p_{x1} nor p_{x2} is conserved. However, if we again rewrite the Lagrangian in terms of a center of mass coordinate $x_G = (m_1x_1 + m_2x_2)/(m_1 + m_2)$, and a separation coordinate $x_s = x_2 - x_1$, the Lagrangian again becomes

$$\mathcal{L}(x_G, x_s, \dot{x}_G, \dot{x}_s) = \frac{1}{2}(m_1 + m_2)\dot{x}_G^2 + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}x_s^2 - \frac{1}{2}k(x_s - l)^2.$$

With these new coordinates, the generalized momentum $p_{xG} = (m_1 + m_2)\dot{x}_G$ is the total momentum and, since the Lagrangian does not depend on x_G, we know immediately that it will be conserved during the motion. The key point here is that the system is invariant under translation, but only if you translate the whole system; if you just translate one of the masses, you stretch the spring and the system isn't invariant at all. Using the center of mass as a coordinate allows us to translate the whole system by changing just one coordinate, and therby expose this invariance, and find the corresponding conserved quantity. Here we have encountered some truly profound theoretical physics. The fact that conducting a mechanical experiment (e.g. swinging a pendulum) at two different points in space produces the same motion is the fundamental reason why total momentum is conserved

5.4 Better conservation of total momentum (non examinable)

The above derivation is rather clunky, as it requires an awkward and specific coordinate choice to find the conservation rule. Having understood that the key point is invariance under translating the whole system, we can exploit this invariance directly by formulating it as a mathematical statement: $\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \mathcal{L}(x_1 + a, x_2 + a, \dot{x}_1, \dot{x}_2)$ .This statement holds for any a, but if a is small we can represent the right side as a Taylor series:

$$\mathcal{L}(x_1 + a, x_2 + a, \dot{x}_1, \dot{x}_2) = \mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) + \sum_{i=1,2}\frac{\partial \mathcal{L}}{\partial x_i}a \implies \sum_{i=1,2}\frac{\partial \mathcal{L}}{\partial x_i}a = 0.$$

If we substitute in the equation of motion $\frac{d}{dt}(p_{xi}) = \frac{\partial \mathcal{L}}{\partial x_i}$ into both terms of the sum, we get

$$\sum_{i=1,2}\frac{\partial \mathcal{L}}{\partial x_i}a = 0 \implies \sum_{i=1,2}\frac{d}{dt}(p_{xi}) = 0 \implies \frac{d}{dt}\sum_{i=1,2}p_{xi} = 0,$$

where this last equation is the equation of conservation of total momentum:

$$\frac{d}{dt}\sum_{i=1,2} p_{xi} = \frac{d}{dt}\sum_{i=1,2} m_i \dot{x}_i = 0.$$

The beauty of this argument is that it trivially generalizes to N particles, m dimensions and different types of invariance. For example, the exact same argument can also be applied to angles and rotations rather than translations. In this case, we discover that angular momentum around an axis will be conserved if the system is "invariant" (i.e. behaves the same way) after a rotation around that axis. In the Universe as a whole, angular momentum is conserved because space is isotropic and linear momentum is conserved because space is translationally invariant.

5.5 Conservation of Energy

In general, if we have a Lagrangian of the form $\mathcal{L}(q_i, \dot{q}_i, t)$, the generalized momentum p_i is conserved if $\mathcal{L}$ does not depend explicitly on q_i, i.e. if $\frac{\partial \mathcal{L}}{\partial q_i} = 0$. Conservation of energy is more complicated, because energy is never one of the conjugate momenta. However, just as x momentum is conserved if $\mathcal{L}$ doesn't depend on x, it can be shown that the total energy

$$E = T + V$$

will be conserved if $\mathcal{L}(q_i, \dot{q}_i, t)$ does not depend (explicitly) on time, i.e if. $\frac{\partial \mathcal{L}}{\partial t} = 0$. All the Lagrangians we have seen so far have this property, but we will see some explicitly time dependent Lagrangians in the last section of the course. This result means that the fact that we get the same motion from a pendulum whether we start it now, or some time later, is the ultimate reason energy is conserved. This result is extremely useful, and we will use it repeatedly, but we will not prove it this year.

6 One degree of freedom

We now turn to solving Lagrangian equations of motion, to actually predict motion. The simplest case is if a mechanism only has a one degree of freedom, q, in which case the Lagrangian will (almost certainly) be of the form:

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - V(q).$$

This Lagrangian gives rise to a single generalized momentum and force

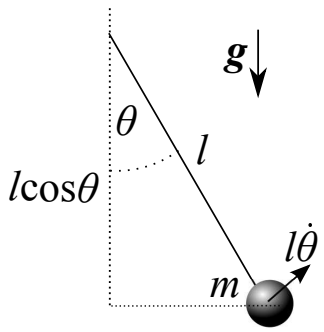
$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} \qquad F = \frac{\partial \mathcal{L}}{\partial q} = -V'(q),$$

so the equation of motion is simply

$$\frac{d}{dt}\left(m\dot{q}\right) = m\ddot{q} = -V'(q).$$

The classic example, which exhibits all the key behaviors of one d.o.f. (degree of freedom) systems, is, once again, the simple pendulum:

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta,$$



for which the generalized momentum and force are

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad F_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta,$$

and the equation of motion is the famous pendulum equation:

$$ml^2\ddot{\theta} = -mgl\sin(\theta), \qquad \Longrightarrow \qquad \ddot{\theta} = -\frac{g}{l}\sin(\theta).$$

In this section, we will give a *complete and final* treatment of the simple pendulum, as an illustration of what one can find in one-degree-of-freedom Lagrangian dynamics.

6.1 First integral of the equation of motion

To find the motion of the system, we need to integrate twice, $\ddot{\theta} \to \dot{\theta} \to \theta$.

The first integration can always be done analytically. The Lagrangian doesn't have any t dependence, so the motion will conserve total energy,

$$E = T + V = \frac{1}{2} m \dot{q}^2 + V(q).$$

In this simple one d.o.f. case, we can easily verify energy conservation from the equation of motion. The trick is to multiply the equation of motion by $\dot{q}$, which allows us to write it as an exact time derivative:

$$\begin{aligned} m\ddot{q}\dot{q} + V'(q)\dot{q} &= 0 \\ \Longrightarrow \frac{d}{dt}\left(\frac{1}{2} m \dot{q}^2 + V(q)\right) &= 0. \end{aligned}$$

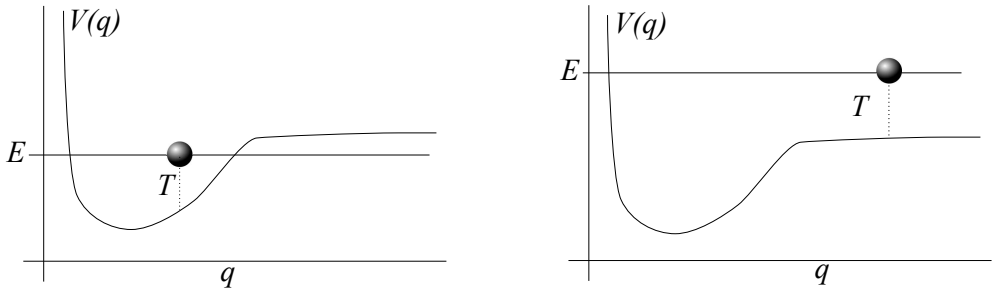
This equation is now a conservation law for the quantity $\frac{1}{2}m\dot{q}^2 + V(q)$, which is E. Integrating once with respect to time gives:

$$\frac{1}{2}m\dot{q}^2 + V(q) = E \quad \text{is constant.}$$

The original equation gave us acceleration as a function of position, but having integrated we now have velocity as a function of position:

$$\dot{q} = \sqrt{\frac{2}{m}\left(E - V(q)\right)}.$$

This result tells us pretty much everything about the motion of the system. If we draw a graph of $V(q)$, we can represent the total energy of a motion, E, as a straight horizontal line. At any position, q, the gap between the line and the potential tells is the kinetic energy of the particle, T from which we can get the velocity $\dot{q}$.

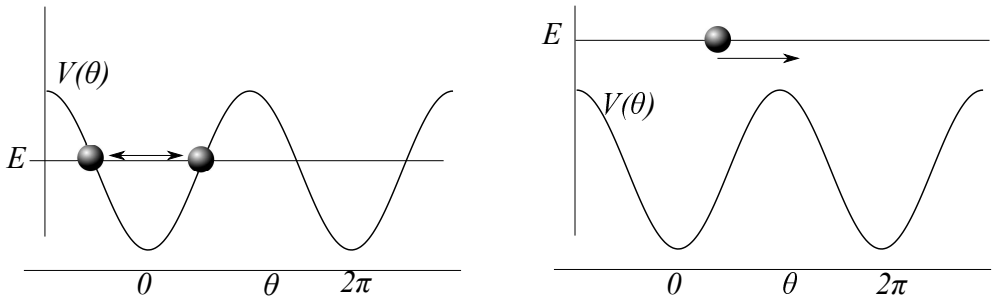


Fundamentally, there are two possibilities. Either the E line intersects the V curve twice (left, above), in which case these are turning points where $\dot{q} = 0$, and the mechanism will move between the two turning points periodically forever. Alternativley, E might be above the V curve (or intersect only once) (right, above) in which case q will never turn round, but instead eventually reach infinity.

For the pendulum case, the first integral (energy conservation) gives us the following $\dot{\theta}$ as a function of position:

$$\dot{\theta} = \sqrt{2\left(\frac{E}{ml^2} + \frac{g}{l}\cos\theta\right)}.$$

Drawing the $V(\theta)$ curve for the pendulum potential $V(\theta) = -mgl\cos(\theta)$, we see the pendulum can show both behaviors, depending on the value of E.



Physically, if $E < mgl$ the pendulum has insufficient energy to reach the vertical upwards configuration, and swings backwards and forwards periodically forever: this is the hallmark oscillatory motion of a pendulum. If $E > mgl$, the pendulum can reach the vertically upwards and swings in circles, winding θ up to larger and larger numbers.

6.2 Equilibrium States

For certain values of the energy, and certain positions of the mechanisms there may be equilibrium states where the mechanisms simply sits still for ever. In general, to find equilibrium points, simply set $q = q_0$ and $\dot{q} = \ddot{q} = 0$ in the equation of motion, and see if there are any solutions. Setting $\ddot{q} = 0$ in the original equation of motion, we see that such a point must have

$$V'(q_0) = 0,$$

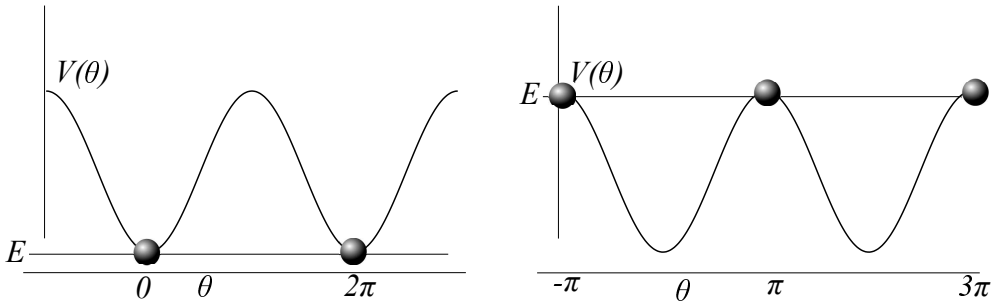
i.e. it must be an stationary point of the potential energy $V(q)$. However, it isn't sufficient just to be at q_0, we also need that $\dot{q} = 0$ which, from the energy expression, tells us that

$$E = V(q_0).$$

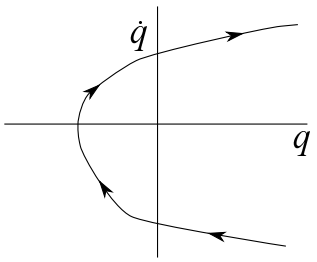
This all makes good intuitive sense in the potential-energy picture: these points occur when the system is at a minima/maxima in the potential energy $V'(q_0) = 0$, and are simultaneously not moving so $E = V(q_0)$. Since the mechanism is at a stationary point it will never start moving, and since it is not moving, it will stay there for ever.

In the case of the pendulum, there are two such equilibrium states, the pendulum can be vertically up or vertically down.

$$\theta = 0,\ \dot{\theta} = 0 \quad \text{and} \quad \theta = \pi,\ \dot{\theta} = 0.$$



6.3 Phase portrait

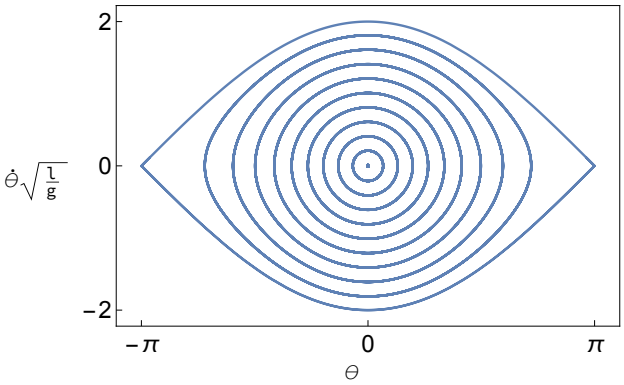


The above ideas can be summarized in one beautiful and informative plot, known as a phase portrait. This is a plot with q on one axis and $\dot{q}$ on the other axis, so each point corresponds to the mechanism being in a certain configuration and moving at certain rate. As the mechanism moves, both q and $\dot{q}$ will change, tracing out a curve for the motion, which is known as an *orbit*. A simple example is shown on the right. Note that the plot is self consistent, as q increases along the trajectory when $\dot{q}$ is positive and decreases when $\dot{q}$ is negative. For this reason, we often don't bother with the arrows indicating direction.

In general there is one such line on the phase portrait for each value of the total energy E, which plots the motion of the mechanism when it moves with that particular value of E.

For the pendulum, the simplest orbit is $E = -mgl$, which is simply the equilibrium point $\theta = \dot{\theta} = 0$. In this "motion" the pendulum never actually moves, so the curve is a point at $\theta = \dot{\theta} = 0$.

For $-mgl < E < mgl$ the pendulum swings backwards and forwards in a periodic motion. These swings form closed loops in the phase portrait, as the motion returns to its starting point each swing. The curves are loops rather than arcs because the pendulum has a different sign of $\dot{\theta}$ on the backward and forward part of the swing. These loops are nested, with lower energy loops inside the higher energy loops, as the higher energy loops have higher amplitude.

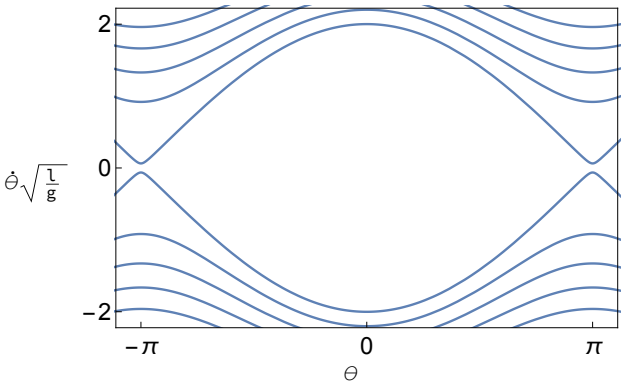


Periodic motions of the pendulum with $E < mgl$.

If E is only slightly smaller than mgl, the pendulum almost reaches the vertical position, forming the largest loop corresponding to the highest amplitude periodic motion.

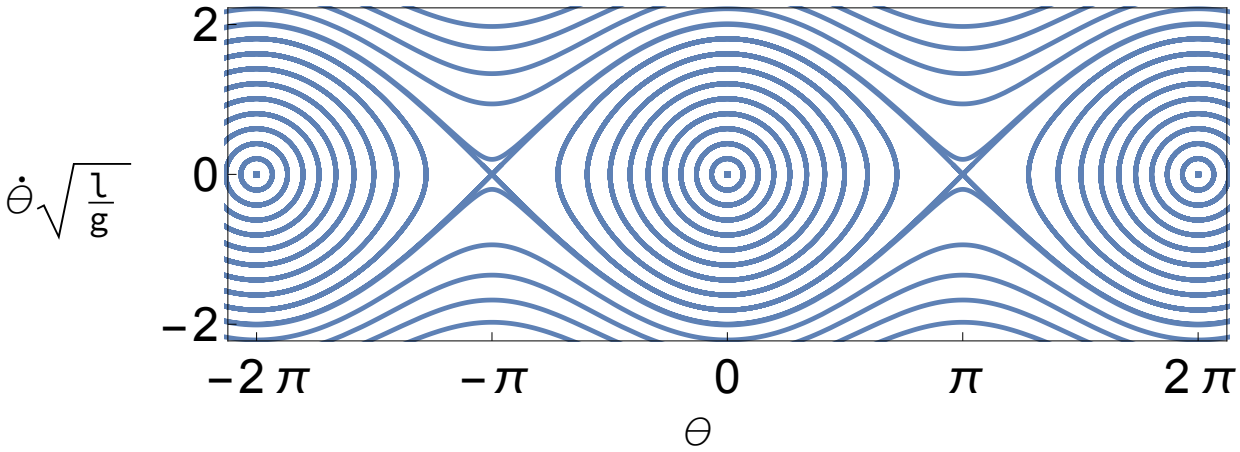
If $E = mgl$ the pendulum sits at the equilibrium point at $\theta = \pm\pi$, $\dot{\theta} = 0$ and the curve is a point.

If $E > mgl$ the pendulum swings in circles winding θ to higher and higher values. This produces infinite lines on the phase portrait rather than closed loops. The pendulum moves fastest at $\theta = 0$ and slower at $\theta = \pm\pi$ where the potential energy is higher.



Circular motions of the pendulum with $E > mgl$.

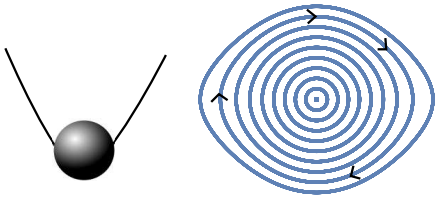
Putting all these different types of loop on one portrait, we get a beautiful summary of all the possible motions of the pendulum. It is helpful to extend the range of θ beyond $\pm\pi$ so we can see more clearly the motion around the $\theta = \pi$ equilibrium.



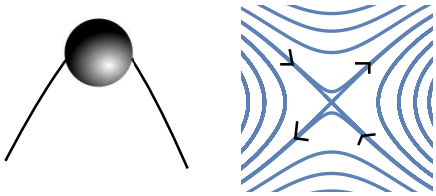
The key reason phase portraits are intelligible and useful is that the lines for different values of E can never cross each other. If they did, and you started the pendulum with the θ and $\dot{\theta}$ of the crossing point, it wouldn't know where to go next!

6.4 Stability and Instability

The two equilibria for the pendulum are physically very different. If the pendulum is given a little push from $\theta = 0$, it must gain potential energy, so it slows down, stops and turns round. It will then swing back and forth periodically, staying close to the $\theta = 0$ equilibrium, and if we added in damping, it would eventually settle back down to $\theta = 0$. This is evident on the phase portrait because the curves around the equilibrium point are loops that enclose it. We call such an equilibrium a stable equilibrium.



If we try the same at $\theta = \pi$, the pendulum will gain kinetic energy as it moves away from the equilibrium and hence head away even faster. The resulting motion will not stay around $\theta = \pi$, and if we add in damping it will settle down to $\theta = 0$. We can see on the phase portrait that even the curves very near the $\theta = \pi$ equilibrium ultimately run far away from it. We call this state an unstable equilibrium.



To probe this mathematically, we consider what happens if our system is a small distance from an equilibrium point, $q = q_0 + \delta q$. If we substitute this into the equation of motion, we get:

$$m\ddot{\delta q} = -V'(q_0 + \delta q).$$

However, since δq is small, we can Taylor expand the right hand side and take advantage of the fact that, at an equilibrium $V'(q_0) = 0$. This gives us

$$m\ddot{\delta q} \approx -V''(q_0)\delta q,$$

which is an equation of motion for δq with very different character depending on the sign of $V''(q_0)$. In both cases the "force" is proportional to displacement δq, but if $V(q_0)$ was a minimum,

$V''(q_0)$ is positive, the force pulls back to equilibrium and this is the equation of simple harmonic motion. For example, for the pendulum, around $\theta = 0 + \delta\theta$ we have

$$\ddot{\delta\theta} = -\frac{g}{l}\delta\theta \implies \delta\theta = a_0 \cos\left(\sqrt{\frac{g}{l}}t + \phi\right),$$

which is the motion of the pendulum vibrating around the equilibrium position.

On the other hand, if $V(q_0)$ was a maximum, $V''(q_0)$ is negative and the force pushes away from the equilibrium. In this case the equation has exponentially growing and decaying solutions. For example, for the pendulum, around $\theta = \pi + \delta\theta$ we have

$$\ddot{\delta\theta} = \frac{g}{l}\delta\theta \implies \delta\theta = Ae^{-\sqrt{\frac{g}{l}}t} + Be^{\sqrt{\frac{g}{l}}t}.$$

The exponentially growing term here corresponds to the system speeding up as it moves away from the equilibrium state. The decaying solution corresponds to launching the pendulum upwards towards the equilibrium point with exactly the right amount of energy to come to rest when it gets there, which is much less likely in practice.

There is an important difference between the two solutions. In the stable case, if $\delta\theta$ starts small it remains small, so the approximate equation of motion remains valid. If the unstable case, even if $\delta\theta$ starts small, it grows until it gets big, at which point the Taylor series approximation breaks down. This doesn't mean the solution is useless: it is very useful because it tells us the growth rate of a small perturbation: i.e. how long you have before a small perturbation grows big.

6.5 Second integral of the equation of motion

Although the phase portrait and the stability analysis give a detailed picture of the dynamics of a pendulum, we still haven't achieved a final answer for the angle of the pendulum as a function of time, $\theta(t)$. We need to integrate a second time, $\dot{\theta} \to \theta$, to get position from velocity. In principle this is easy. We have an expression for the velocity from the first integral (energy conservation):

$$\dot{q} = \sqrt{\frac{2}{m}\left(E - V(q)\right)}.$$

If we divide through by the right hand side then integrate with respect to time, we have a q integral on the left and a t integral on the right:

$$\int \frac{\sqrt{m/2}}{\sqrt{E - V(q)}} \mathrm{d}q = \int \mathrm{d}t = t.$$

This means, if we start from q_i, and want to know when the particle reaches q_f, all we have to do is work out the definite integral

$$t_{q_i \to q_f} = \int_{q_i}^{q_f} \frac{\sqrt{m/2}}{\sqrt{E - V(q)}}\, \mathrm{d}q.$$

All energy conserving one degree of freedom motion can thus be reduced to a single final integral. We say we the motion has been *reduced to quadrature*. Sadly, the integral is rarely tractable analytically, so now we need a computer.

6.5.1 Second integral for the simple pendulum (non-examinable)

For the case of the pendulum, if we release it at rest at an angle θ_0, it will have energy $E = -mgl\cos(\theta_0)$, so it will reach θ at a time given by:

$$t_{\theta_0\to\theta} = \int_{\theta_0}^{\theta} \frac{\sqrt{ml^2/2}}{\sqrt{E+mgl\cos\theta}}\,\mathrm{d}\theta = \sqrt{\frac{l}{2g}}\int_{\theta}^{\theta_0}\frac{1}{\sqrt{\cos\theta-\cos\theta_0}}\,\mathrm{d}\theta.$$

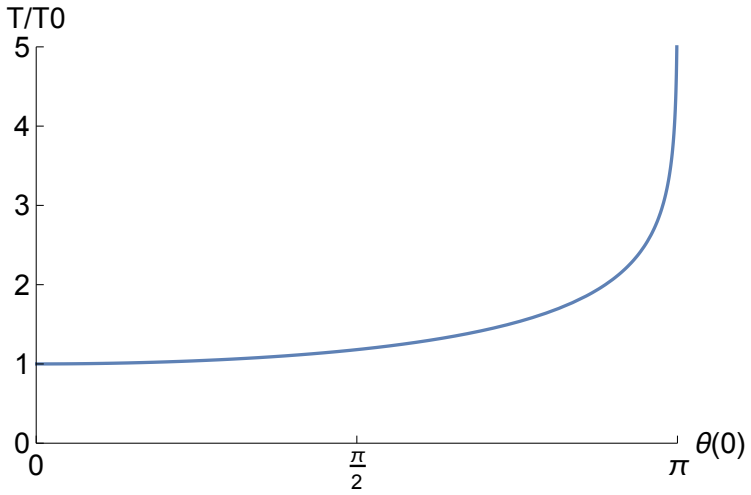
To calculate the time period of the pendulum's oscillations T (sorry again) then we can work it out as four times the length of time it takes to get from θ_0 to $\theta = 0$:

$$T = 4\sqrt{\frac{l}{2g}}\int_{0}^{\theta_0}\frac{1}{\sqrt{\cos\theta-\cos\theta_0}}\,\mathrm{d}\theta.$$

This integral doesn't have an answer in terms of analytic functions, but it is easy enough to evaluate on a computer. If you want to look clever, you can make a cunning substitution, $\sin u = \frac{\sin(\theta/2)}{\sin(\theta_0/2)}$, which brings the integral into the form:

$$T = 4\sqrt{\frac{l}{g}}\int_{0}^{\pi/2}\frac{1}{\sqrt{1-\sin^2(\theta_0/2)\sin^2 u}}\,\mathrm{d}u \equiv T_0\frac{2}{\pi}K\left(\sin\frac{\theta_0}{2}\right)$$

where $K(x)$ is a special function called the "complete elliptic integral of the first kind," and $T_0 = 2\pi\sqrt{\frac{l}{g}}$ is the SHM time period of a small amplitude oscillations. However, if you look up what the definition of $K(x)$ is, you will find it is defined as this integral, so this hasn't really moved us much further on, except we can now evaluate the function on a serious calculator. This allows us to make a nice plot of the pendulum time period as a function of amplitude.



Notably, the time period diverges as $\theta_0 \to \pi$, which makes sense as, in this limit, the pendulum has just enough energy to reach vertical, but slows down a great deal as it gets there, and almost reaches the vertical equilibrium point before falling back down again.

We can also look up the Taylor series of $K(x)$ for small x to make a series expansion for the time period of a pendulum in its amplitude:

$$T = T_0\left(1+\frac{\theta_0^2}{16}+\frac{11\theta_0^4}{3072}+\frac{173\theta_0^6}{737280}+\frac{22931\theta_0^8}{1321205760}+\ldots\right).$$

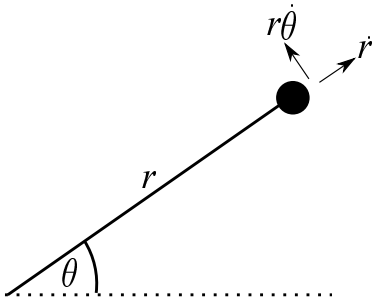
One word of warning, different sources define $K(x)$ in different ways. Some authors and programming languages (including wolfram-alpha) use a convention where $K(x) \to K(x^2)$.

7 Effective potential

Although we can tackle one-degree-of-freedom problems with Lagrangians, the real benefit lies with more complicated systems. Unfortunately, in these cases, the equations themselves are often complicated, coupled non-linear differential equations which are very difficult to solve. For the rest of the course, we will be looking at strategies for understanding the motion of these more complicated systems.

One helpful strategy is to use conservation laws (typically conservation of angular momentum) to effectively reduce the number of degrees of freedom for a system. If we can turn the problem into a one-degree-of-freedom problem, we can then exploit our insights from the previous section.

7.1 Motion in a central potential



This strategy is very helpful when we have a particle moving in a potential energy which, in polar coordinates, takes the form $V(r)$. Such a potential is called a "central potential," and arises naturally whenever the only force on the particle is back towards the origin. Important examples include movement of a satellite around the Earth ($V(r) = -GMm/r$), motion of an electron around an atom, or motion of a particle tethered to the origin with a spring or string ($V(r) = \frac{1}{2}k(r-l)^2$). In all these case, taking the coordinates of the particle as (r, θ), the Lagrangian has the form:

$$\mathcal{L}(r, \theta, \dot{r}, \dot{\theta}) = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r).$$

The generalized momenta are,

$$p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}, \qquad p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta},$$

which are the actual momentum in the r direction and the angular momentum around the origin. There is no θ dependence in the Lagrangian, so $F_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = 0$, and the equation of motion for θ simply says that angular momentum, h, will be conserved:

$$\frac{d}{dt}\left(p_\theta\right) = 0 \implies mr^2\dot{\theta} = h.$$

This is expected because the problem is rotationally invariant. In more Newtonian terms, the only force on the particle always acts through the origin so there is no torque in the z direction.

In contrast, there is r dependence in $\mathcal{L}$ so the generalized force is not zero but rather:

$$F_r = \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\theta}^2 - V'(r)$$

and the equation of motion for r is

$$m\ddot{r} = mr\dot{\theta}^2 - V'(r).$$

We now have two equations of motion for two variables, θ and r, and our task is to solve these equations to find the motion. Fortunately, we have already integrated the θ equation once to get a simple conservation rule for angular momentum. This rule tells us that, during the motion, $\dot{\theta}$ is determined by the radius:

$$\dot{\theta} = \frac{h}{mr^2}.$$

If we substitute this into the equation of motion for r, we can eliminate θ and $\dot{\theta}$ entirely:

$$m\ddot{r} = \frac{h^2}{mr^3} - V'(r).$$

We now have a one degree of freedom equation to solve for $r(t)$. It is in the standard form for 1D motion in a potential, $m\ddot{r} = -V'_{eff}(r)$, with a modified *effective potential* energy $V_{eff}(r)$, where

$$V_{eff}(r) = \frac{h^2}{2mr^2} + V(r).$$

The additional term, which is called the centrifugal potential, $V_{cent}(r)$. If we re-insert the definition of h, we see that ,

$$V_{cent}(r) = \frac{h^2}{2mr^2} = \frac{1}{2}mr^2\dot{\theta}^2,$$

i.e. it is the kinetic energy of the particle associated with its motion in the θ direction.

This observation makes the centrifugal potential quite intuitive. As the particle approaches the origin $\dot{\theta}$ has to increase to conserve angular momentum, increasing the kinetic energy associated with $\dot{\theta}$. This leads to a kinetic energy barrier as the particle approaches the origin, which grows as $1/r^2$: the particle would need an infinite $\dot{\theta}$, and hence infinite kinetic energy, to reach the origin whilst conserving angular momentum.

As with all one-degree-of-freedom systems, we can now multiply the equation of motion by $\dot{r}$ and integrate to get a first integral,

$$\frac{1}{2}m\dot{r}^2 + V_{eff}(r) = E,$$

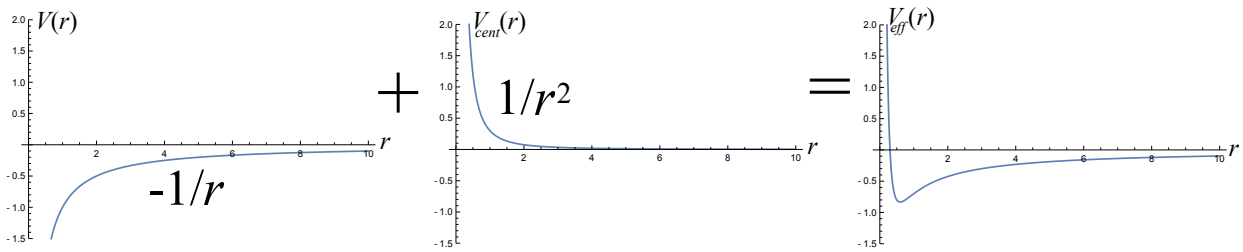
where E is a constant of integration corresponding to the total energy.

7.1.1 Satellite Motion

For a satellite going around Earth, the potential has the form $V = -GMm/r \equiv -A/r$, so the effective potential takes the form:

$$V_{eff}(r) = \frac{h^2}{2mr^2} - \frac{A}{r}.$$

If we sketch this potential as a function of r, we see that $-1/r$ dominates as $r \to \infty$, but $+1/r^2$ dominates as $r \to 0$, so the potential always diverges positively near the origin and tends to zero from below at infinity and has a has a single minimum in between.



The radial behavior of the satellite is just like a 1D motion in this new potential. There is one equilibrium point at the minimum, which is located at

$$V'_{eff}(r_{min}) = 0 \implies -\frac{h^2}{mr_{min}^3} + \frac{A}{r_{min}^2} = 0 \implies r_{min} = \frac{h^2}{mA}.$$

If the satellite has just the right amount of energy, it can sit at this minimum. Of course, the satellite isn't actually sitting still, it just has a constant radius — i.e. the satellite is going in circles. Its angular velocity is:

$$\dot{\theta} = \frac{h}{mr_{min}^2} = \frac{\sqrt{mAr_{min}}}{mr_{min}^2} = \sqrt{\frac{A}{mr_{min}^3}},$$

as we would have got from A-level "motion is a circle" formula $-A/r^2 = mr\dot{\theta}^2$. Although the value of h is fixed by during a given satellite motion, and hence the radius r_{min} is fixed during a given motion, you can of course give the satellite a different h if you start its with a different angular momentum and, for that motion, the location of r_{min} will be different — this is why, in practice, satellites can have circular orbits at any radius we need.

If the satellite has a little more energy, it will oscillate back and forth periodically in the potential minimum. The circular orbit is stable, because it is at a minimum in the effective potential. In terms of the real orbit, this radial vibration amounts to going around in a circle with slightly undulating radius... i.e. something like an ellipse.

For small oscillations we can do a small vibration analysis to find the frequency of these undulations. Setting $r = r_{min} + \delta r$, we can Taylor expand the effective potential to get:

$$V_{eff}(r_{min} + \delta r) \approx V_{eff}(r_{min}) + \frac{1}{2}V''_{eff}(r_{min})\delta r^2.$$

Substituting this into the equation of motion, we get the expected SHM equation and solution:

$$m\ddot{\delta r} = -V''_{eff}(r_{min})\delta r \qquad \implies \qquad \delta r = a_0 \cos\left(\sqrt{\frac{V''_{eff}(r_{min})}{m}}t + \phi\right).$$

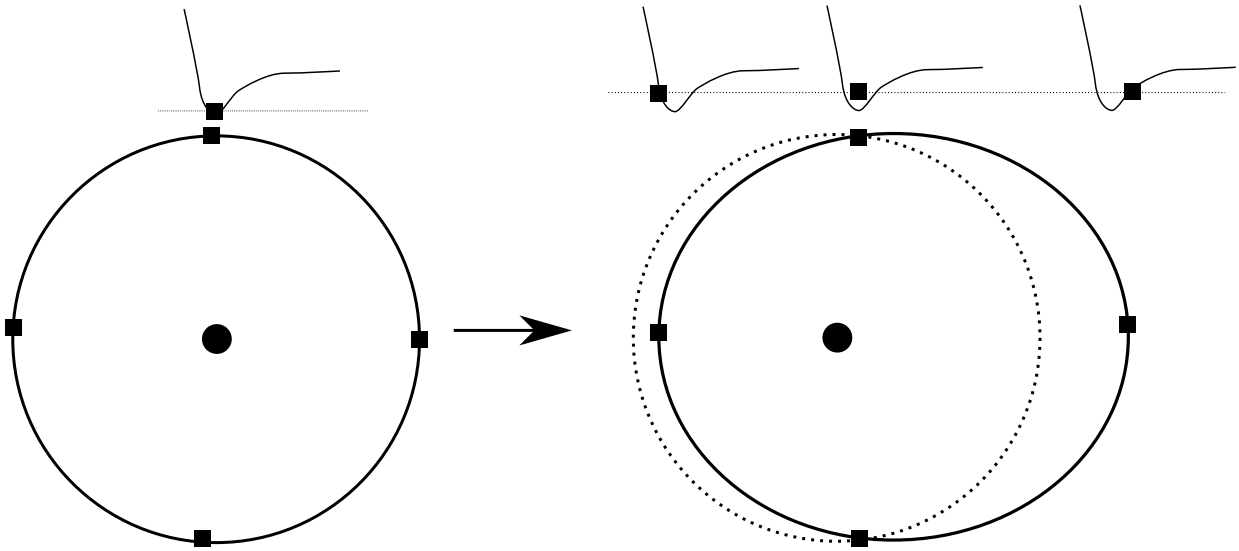
To calculate the time period' we need to evaluate $V''_{eff}(r_{min})$. First taking the second derivative gives:

$$V''_{eff}(r) = \frac{3h^2}{mr^4} - \frac{2A}{r^3},$$

secondly, inserting our result for $V'(r_{min}) = 0$ to eliminate h,

$$V''_{eff}(r_{min}) = \frac{3Ar_{min}}{r_{min}^4} - \frac{2A}{r_{min}^3} = \frac{A}{r_{min}^3}.$$

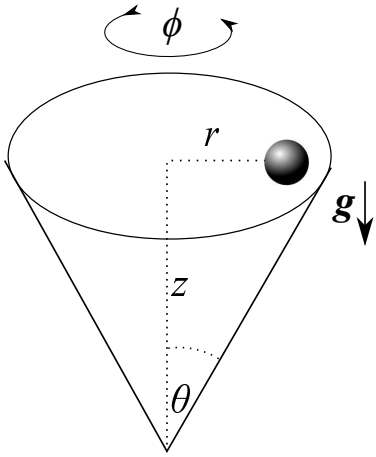
The angular frequency for these radial vibrations, $\sqrt{\frac{A}{mr_{min}^3}}$, is the same as the angular velocity of the underlying circular orbit. This means that the orbit does exactly one complete radial undulation on each circular orbit, so the orbit *closes* (i.e. gets back to where it started) and forms a focus centered ellipse In the figure above, we can see how the radial vibration in the effective potential changes a circular orbit into an elliptical one.



If the satellite has even more energy, the quadratic small-vibrations analysis will not apply, but we know from last year the orbit remains closed and elliptical.

If the total energy increases above zero, the 1-D radial motion will not be bound and periodic, but escape to infinity. Five man-made objects have achieved this state in the solar system: Voyager 1, Voyager 2, New Horizons, Pioneer 10 and Pioneer 11.

7.1.2 Motion in a cone



Consider again a particle sliding along the surface of a cone under gravity. We already calculated the Lagrangian:

$$\mathcal{L}(r,\phi,\dot{r},\dot{\phi}) = \frac{1}{2}m\left(\dot{r}^2\csc^2\theta + r^2\dot{\phi}^2\right) - mgr\cot\theta.$$

We now understand that this is an example of motion in a central potential $V(r) = mgr\cot\theta$, which is linear in r. The generalized momenta and forces are:

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mr^2\dot{\phi} \qquad F_\phi = \frac{\partial\mathcal{L}}{\partial\phi} = 0$$
$$p_r = \frac{\partial\mathcal{L}}{\partial r} = m\dot{r}\csc^2\theta \qquad F_r = \frac{\partial\mathcal{L}}{\partial r} = mr\dot{\phi}^2 - mg\cot\theta.$$

The equation of motion for ϕ is indeed conservation of z angular momentum, h:

$$\frac{d}{dt}(p_\phi) = 0 \qquad \Longrightarrow \qquad mr^2\dot{\phi} = h,$$

and the equation of motion for r is

$$\frac{d}{dt}(p_r) = m\ddot{r}\csc^2\theta = mr\dot{\phi}^2 - mg\cot\theta \qquad \Longrightarrow \qquad m\ddot{r} = mr\dot{\phi}^2\sin^2\theta - mg\cos\theta\sin\theta.$$

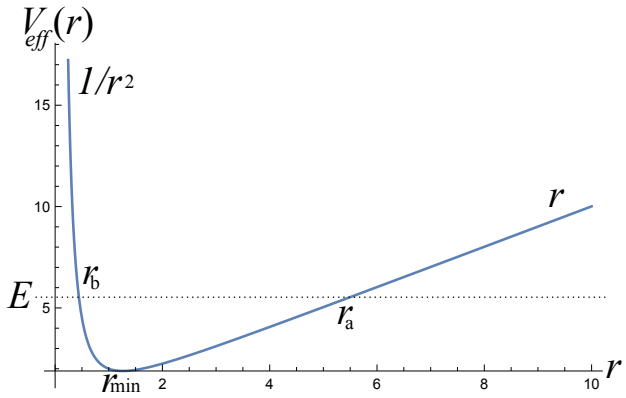
Again using angular momentum conservation $mr^2\dot{\phi} = h$ to eliminate $\dot{\phi}$ from this equation,

$$m\ddot{r} = \frac{h^2\sin^2\theta}{mr^3} - mg\cos\theta\sin\theta$$

we see it is in the form of 1-D motion in an effective potential

$$V_{eff}(r) = \frac{h^2\sin^2\theta}{2mr^2} + mgr\cos\theta\sin\theta,$$

the main difference from the satellite case is that the new $V_{eff}(r)$ grows linearly for large r, rather than asymptoting to zero. The effective potential has one minimum, corresponding again

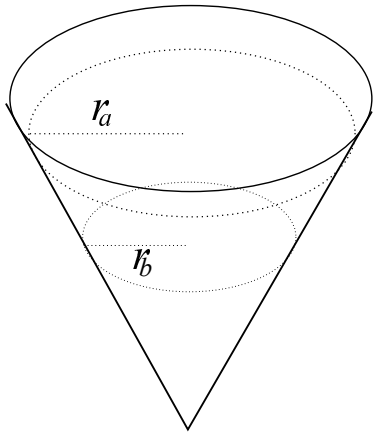


to movement in circles around the cone at radius r_{min}, where

$$V'_{eff}(r_{min}) = 0 \implies r_{min}^3 = \frac{h^2\tan\theta}{gm^2},$$

and the angular velocity of these circles is

$$\dot{\phi} = \frac{h}{mr_{min}^2} = \sqrt{\frac{g}{r_{min}\tan\theta}}.$$



If the particle has more energy that $V_{eff}(r_{min})$, as shown in the plot above, we will have bound periodic motion of r between two radii r_a and r_b, meaning the actual motion is bounded by an upper and lower circle on the cone. Given the potential diverges for $r \to 0$ and $r \to \infty$, there are no unbound motions. As with the pendulum, we cannot compute the second integral analytically for a full closed solution for motion in a cone. However, if the motion is a small perturbation from the circular orbit $r = r_0 + \delta r$, we can again use a small vibration analysis to show that it will perform a simple harmonic motion of the form

$$m\ddot{\delta r} = -V''_{eff}(r_{min})\delta r \qquad \Longrightarrow \qquad \delta r = a_0\cos\left(\sqrt{\frac{V''_{eff}(r_{min})}{m}}t + \phi\right).$$

To evaluate the frequency of the oscillation, we need

$$V''(r) = \frac{3h^2 \sin^2\theta}{mr^4},$$

which, if we substitute in $r_{min}^3 = \frac{h^2 \tan\theta}{gm^2}$ for h gives us

$$V''(r_{min}) = \frac{3gm\cos\theta\sin\theta}{r_{min}},$$

so the angular frequency of the radial vibration

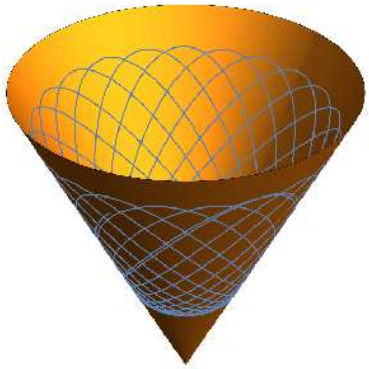
$$\sqrt{\frac{1}{m}V''(r_{min})} = \sqrt{\frac{3g\cos\theta\sin\theta}{r_{min}}}$$

At the magic cone angle the motion closes.

differs from the angular velocity of the underlying circular orbit by a factor of $\sqrt{3}\sin\theta$. For one magic cone angle,

$$\theta = \sin^{-1}\frac{1}{\sqrt{3}} \approx 35\,\text{deg}$$

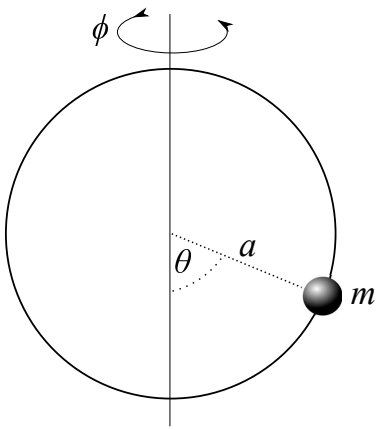
this ratio is one and the orbit closes. For a cone close to this value, the orbit will nearly close, but get back to the original ϕ value just before/after getting back to the original r value. This creates an orbit that *precesses* in time, with the ellipse of the orbit rotating a little between revolutions. A motion like this, with two periodic components that have incommensurate frequencies never perfectly repeats, and is called *quasiperiodic*.



At other cone angles the motion precess.

Notice that, although we haven't been able to solve the motion completely, we have been able to capture its key features through a cunning combination of stability analysis and effective potential.

7.2 Bead on a freely rotating hoop



The effective potential idea is not limited to central potentials: it is often useful in other situations where angular momentum (or indeed some other quantity) is conserved. For example, consider a bead that can slide around a hoop, which in turn is free to spin around a diameter, as shown on the right.

Unlike last time we looked at a bead on a spinning hoop, this is a two-degree of freedom mechanism, since the hoop is free to rotate rather than being spun at fixed angular velocity. However, like last time, the Lagrangian is entirely kinetic energy:

$$\mathcal{L}(\theta, \phi, \dot{\theta}, \dot{\phi}) = \frac{1}{2} m \left(a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \dot{\phi}^2 \right).$$

The generalized momenta and forces are

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ma^2 \dot{\theta} \qquad F_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = ma^2 \sin\theta \cos\theta \dot{\phi}^2$$
$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ma^2 \sin^2 \theta \dot{\phi} \qquad F_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

As we expect, $\mathcal{L}$ does not depend on ϕ, so the equation of motion for ϕ says that p_ϕ is conserved:

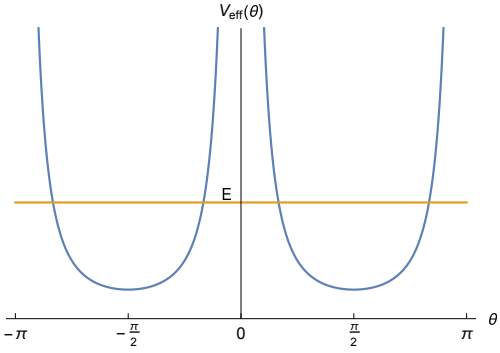
$$\frac{d}{dt}\left(ma^2 \sin^2 \theta \dot{\phi} \right) = 0 \qquad \Longrightarrow \qquad ma^2 \sin^2 \theta \dot{\phi} \equiv h \quad \text{is constant.}$$

The total energy is then

$$E = \frac{1}{2} ma^2 \dot{\theta}^2 + \frac{h^2}{2ma^2 \sin^2 \theta},$$

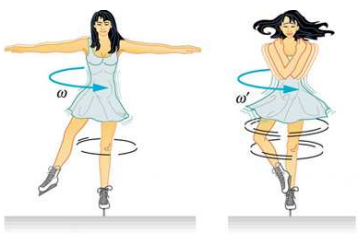
so the θ motion of the particle is equivalent to a body with moment of inertia ma^2 moving in an angular potential

$$V_{eff}(\theta) = \frac{h^2}{2ma^2 \sin^2 \theta}.$$



Angular effective potential.

The minima of this potential is at $\theta = \pm\pi/2$, so the equilibrium solutions are the mass sitting at $\pm\pi/2$ and going in circles.



The effective potential has infinite energy barriers at $\theta = 0, \pi$, so no matter how much energy the bead has, it will bounce back and forth between two intermediate symmetric values of θ staying on one side of the hoop.

As the bead approaches $\theta = 0, \pi$, it also approaches the vertical axis, so the hoop must spin faster to conserve angular momentum. The overall motion is thus a lumpy sort of rotation for the hoop, with the hoop spinning slower when the mass is at $\pi/2$ and faster when it gets closer to the axis. This effect — spinning faster when mass moves towards the rotation axis — is exploited by ice skaters to achieve high spin rates by pulling their arms in.

8 Two body problems

A second case where a complicated problem can be reduced a one-degree of freedom problem is the case of two interacting particles, known as the two body problem.

Consider two particles, m_1 and m_2, with positions $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$. If the two exert force on each other, the potential will be a function of their separation $r = |\mathbf{r}_1 - \mathbf{r}_2|$). For example, if we have two stars, the potential energy would be $-Gm_1m_2/r$, and if we have two masses connected by a spring, the potential will be $(1/2)k(r-l)^2$. The Lagrangian is simply

$$\mathcal{L}\left(\mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2\right) = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|).$$

Following our previous work on conservation of total momentum, we change generalized coordinates to the center of mass $\mathbf{r}_G = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$,and the vector separation of the particles $\mathbf{r}_S = \mathbf{r}_2 - \mathbf{r}_1$. In terms of these new coordinates, the actual particle positions are

$$\mathbf{r}_1 = \mathbf{r}_G - \frac{m_2}{m_1 + m_2}\mathbf{r}_S \qquad \mathbf{r}_2 = \mathbf{r}_G + \frac{m_1}{m_1 + m_2}\mathbf{r}_S.$$

In terms of the new coordinates, we can calculate the kinetic energy exactly as we did in the momentum conservation section

$$T = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 = \frac{1}{2}(m_1 + m_2)|\dot{\mathbf{r}}_G|^2 + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}|\dot{\mathbf{r}}_S|^2,$$

so, in terms of these new coordinates, the Lagrangian is simply:

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)|\dot{\mathbf{r}}_G|^2 + \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}|\dot{\mathbf{r}}_S|^2 - V(|\mathbf{r}_S|).$$

This Lagrangian is now the sum of two independent sub-Lagrangians, one for each coordinate, so the motions of the two coordinates are completely uncoupled. We essentially have two separate Lagrangians:

$$\mathcal{L}_G = \frac{1}{2}(m_1 + m_2)|\dot{\mathbf{r}}_G|^2 \qquad \mathcal{L}_S = \frac{1}{2}\frac{m_1m_2}{m_1 + m_2}|\dot{\mathbf{r}}_S|^2 - V(|\mathbf{r}_S|).$$

The first tells us the equation of motion for the center of mass which, as expected, is the equation of conservation of total momentum:

$$\frac{d}{dt}\left((m_1 + m_2)\dot{\mathbf{r}}_G\right) = 0.$$

This implies that the center of mass moves in a straight line at a constant velocity.

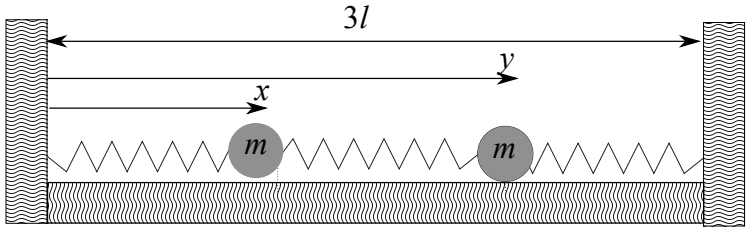
The motion of the separation coordinate is in the general form of a particle moving in a central potential $V(r)$, but with a modified mass $\mu = \frac{m_1m_2}{m_1+m_2}$ known as the reduced mass. If we can solve the problem for a particle moving in a central potential $V(r)$, we can also solve the corresponding two body problem. For example, with gravity, since we can solve the problem of a satellite orbiting Earth (a one body central potential problem) we can also solve the motion of binary stars.

This is quite the achievement. We started with a six degree of freedom problem, $(x_1, y_1, z_1, x_2, y_2, z_2)$, but by carefully explotiting conservation laws (linear momentum, angular momentum and energy) we can reduce the problem to an energy conserving motion in a 1-D effective potential. The three body problem is a whole different kettle of fish: you start with nine degrees of freedom, and you need to use a computer.

9 Stability and Normal Modes

Unfortunately, most complex mechanisms cannot be reduced to Lagrangians with a single degree of freedom, so we need some strategies for for understanding n dimensional motion. Taking inspiration from the one d.o.f case, a good starting point is to find equilibrium states of such mechanisms and then ask what happens if the system is in such a state and you give it a little kick. The motion of many engineering systems is described by small movements around an equilibrium state, and in fact you learnt the first part of the solution from last years multi-degree of freedom mechanical vibrations content.

9.1 Mass and spring example



Before we think about the general case, it is helpful to run thorough a simple example. Consider the 2 d.o.f. system of two masses m and three springs, all with length l and constant k. The Lagrangian for this system is:

$$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(x-l)^2 - \frac{1}{2}k(3l-y-l)^2 - \frac{1}{2}k(y-x-l)^2.$$

The generalized momental and forces are, in this case, the actual momenta and forces:

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}, \qquad F_x = \frac{\partial \mathcal{L}}{\partial x} = -k(x-l) + k(y-x-l)$$
$$p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}, \qquad F_y = \frac{\partial \mathcal{L}}{\partial y} = k(2l-y) - k(y-x-l).$$

The first thing to do is find an equilibrium state of the system. The equations of motion are

$$\frac{d}{dt}(p_x) = F_x \qquad \frac{d}{dt}(p_y) = F_y$$

so, to find an equilibrium state, we need a configuration where these forces vanish. This is two linear equations in two unknowns (x and y), so we expect one solution, which is naturally at $x_0 = l$ and $y_0 = 2l$. In this state all the springs are their natural length, so this is the minimum potential energy state.

We next imagine the system is displaced a small distance from this equilibrium state, and ask how it then moves. To do this mathematically, we set

$$x = l + \delta x, \qquad y = 2l + \delta y,$$

which makes the forces

$$F_x = -2k\delta x + k\delta y$$
$$F_y = -2k\delta y + k\delta x,$$

and the equations of motion

$$m\delta\ddot{x} = -2k\delta x + k\delta y$$
$$m\delta\ddot{y} = -2k\delta y + k\delta x.$$

It isn't immediately obvious how to solve this pair of equations, but, as you learnt last year, the trick is to search for "normal modes" of vibration, in which *all parts of the system vibrate at the same frequency*. Following last years mechanical vibrations course, we store our unknowns in a vector $(\delta x, \delta y)$, and try an oscillatory trial solutions which, using complex notation, looks like:

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}A\\ B\end{pmatrix} \mathrm{e}^{i\omega t} \quad \rightarrow \quad \begin{pmatrix}\delta\dot{x}\\ \delta\dot{y}\end{pmatrix} = i\omega \begin{pmatrix}A\\ B\end{pmatrix} \mathrm{e}^{i\omega t} \quad \rightarrow \quad \begin{pmatrix}\delta\ddot{x}\\ \delta\ddot{y}\end{pmatrix} = -\omega^2 \begin{pmatrix}A\\ B\end{pmatrix} \mathrm{e}^{i\omega t}.$$

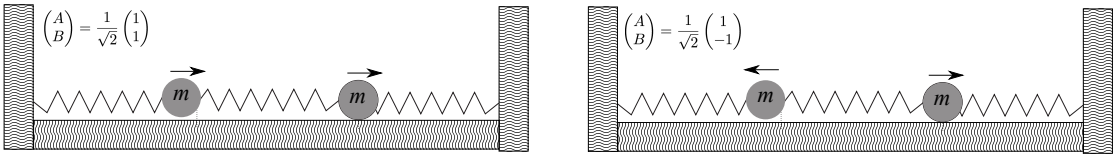
Substituting this solution into the equations of motion, the $e^{i\omega t}$ cancels everywhere, and the differential equations turn into algebraic equations for the constants A, B and ω which have the convenient matrix representation

$$-m\omega^2 \begin{pmatrix}A\\ B\end{pmatrix} = -\begin{pmatrix}2k & -k\\ -k & 2k\end{pmatrix} \begin{pmatrix}A\\ B\end{pmatrix}.$$

This equation actually an eigenvector equation for (A, B), which we can solve by inspection, to find two eigenvectors:

$$\begin{pmatrix}A_1\\ B_1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}, \omega_1^2 = \frac{k}{m} \qquad \begin{pmatrix}A_2\\ B_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}, \omega_2^2 = \frac{3k}{m}.$$

These eigenvectors are solutions to the motion of the system, in which the entire system vibrates at a single frequency — *normal modes of oscillations*. We have found two different modes, with different frequencies and different spatial patterns of vibration. In the first the two masses move together and the central spring is never stretched. In the second, the two masses move in opposite directions, and all three springs are active.



The equations of motion are linear, so we can add (superpose) solutions to get more solutions:

$$\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \frac{C_1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}\cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) + \frac{C_2}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

where I have returned to purely real notation, since the positions of the masses are real quantities. We know this is the most general solution because it contains four constants of integrations, C_1, C_2, ϕ_1 and ϕ_2, as expected for a system of two second order differential equations. The four constants will be determined by four initial conditions: the initial position and velocity of each mass. This means the motion of the system is completely described by a superposition of its two normal modes. The equilibrium point is stable, since if we give it a kick, the system vibrates but remains close to the equilibrium.

9.2 General Normal Mode Treatment

Let us now consider a general n degree of freedom mechanism, with coordinates $q_1, q_2 .... q_n$, which, following our experience in the simple case above, we store in a vector $\mathbf{q} = (q_1, q_2 .... q_n)$. Last year in vibrations, you learnt to formulate the equations of motion, then expand (linearize) them around an equilibrium point, $\mathbf{q} = \mathbf{q}_0 + \delta\mathbf{q}$, identify a mass ($M$) and stiffness ($K$) matrix, and then use an eigenvector approach to find the modes and frequencies. The same approach works with equations of motion from Lagrangian mechanics. However, here we will do something slightly more sophisticated, and expand the Lagrangian itself rather than waiting until we have equations of motion. As a rule of thumb, the earlier you expand the less the algebra, so this approach tends to be a little easier to implement. Returning to our general Lagrangian, since the kinetic energy ultimately comes from $(1/2)mv^2$, it is quadratic in the $\dot{q}_i$, so the kinetic energy will always have the form

$$T = \sum_{i,j} \frac{1}{2} M_{ij}(\mathbf{q}) \dot{q}_i \dot{q}_j = \frac{1}{2} \dot{\mathbf{q}}^T M(\mathbf{q}) \dot{\mathbf{q}},$$

where $M(\mathbf{q})$ is a matrix of mass like quantities with elements M_{ij}. We have kept the possibility of q_i dependence in the "masses" because, for example, in polar coordinates a mass has $T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$, which, in terms of an M matrix is

$$T = \frac{1}{2}(\dot{r}, \dot{\theta}) \begin{pmatrix} m, & 0 \\ 0, & mr^2 \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}.$$

You can (and should) always choose M to be symmetric, as an antisymmetric part would make no contribution to T.

In general, the potential energies will not be quadratic, but it won't depend on velocity, so the Lagrangian of our n d.o.f system will have the form:

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T M(\mathbf{q}) \dot{\mathbf{q}} - V(\mathbf{q}).$$

The generalized force for q_k is potentially complicated because of the q dependence in M:

$$F_k = \frac{\partial \mathcal{L}}{\partial q_k} = \sum_{i,j} \frac{1}{2} \frac{\partial M_{i,j}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial V}{\partial q_k}.$$

We first search for equilibrium states where the system can just sit in a constant configuration with $\mathbf{q} = \mathbf{q}_0$, meaning all the velocities are zero, $\dot{\mathbf{q}} = \mathbf{0}$. For this to occur, we need the generalized forces to all vanish. Fortunately, since we are searching for a state where the velocities are all zero, the first (complicated) term in the generalized force vanishes, so the generalized forces will all vanish if

$$\frac{\partial V}{\partial q_i} = 0$$

for each coordinate, i.e. we need that the configuration is a stationary point in the potential energy.

We next consider giving the system a kick, so that its coordinates are a little removed from the equilibrium state. Mathematically, we write:

$$\mathbf{q} = \mathbf{q}_0 + \delta\mathbf{q}, \qquad \dot{\mathbf{q}} = \delta\dot{\mathbf{q}}, \qquad \ddot{\mathbf{q}} = \delta\ddot{\mathbf{q}}.$$

If we substitute these into the Lagrangian itself, and Taylor expand assuming $\delta\mathbf{q}$ and $\delta\dot{\mathbf{q}}$ are small, we get, to second order:

$$\mathcal{L}(\delta\mathbf{q}, \delta\dot{\mathbf{q}}) \approx V(\mathbf{q}_0) + \frac{1}{2}\delta\dot{\mathbf{q}}^T M \delta\dot{\mathbf{q}} - \frac{1}{2}\delta\mathbf{q}^T K \delta\mathbf{q}$$

where M and K are *constant symmetric matrices* given by:

$$M_{ij} = \left.\frac{\partial^2 T}{\partial\dot{q}_i \partial\dot{q}_j}\right|_{\mathbf{q}=\mathbf{q}_0} = M_{ij}(\mathbf{q}_0), \qquad K_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\mathbf{q}=\mathbf{q}_0}.$$

It is important to unpack why there are no linear terms in the expansion of $\mathcal{L}$. There are no linear $\dot{q}_i$ terms because T itself is quadratic in $\dot{q}_i$. There are no linear q_i terms because we are expanding around a stationary point in the potential energy, $\mathbf{q}_0$.

If we store the full set of generalized momenta and forces in vectors, $\mathbf{p}$ and $\mathbf{F}$, as we did for the coordinates, then, in terms of the new, Taylor expanded, Lagrangian, they evaluate to:

$$\mathbf{p} = \frac{\partial\mathcal{L}}{\partial\delta\dot{\mathbf{q}}} = M\delta\dot{\mathbf{q}}, \qquad \mathbf{F} = \frac{\partial\mathcal{L}}{\partial\delta\mathbf{q}} = -K\delta\mathbf{q},$$

and the full set of equations of motion can be written compactly as

$$M\delta\ddot{\mathbf{q}} = -K\delta\mathbf{q}.$$

To search for normal mode solutions, we try solutions where all the coordinates vibrate at ω,

$$\delta\mathbf{q} = \mathbf{A}e^{i\omega t},$$

where $\mathbf{A} = (A_1, A_2 .... A_n)$ is a constant vector giving the "shape" of the mode. Substituting this into the equation of motion, we get

$$-\omega^2 M\mathbf{A} = -K\mathbf{A} \tag{1}$$

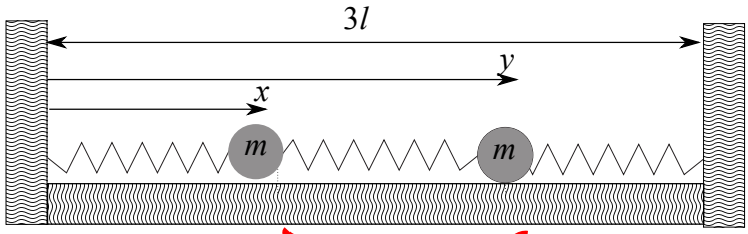
which is known as a *generalized eigenvalue problem* for $\mathbf{A}$ and ω. It is a *generalized* problem because there is a matrix on both sides of the equation. There are two approaches to solving this problem. One is simply to spot solutions. The second, more methodical way, is to note the equation will have solutions if and only if

$$\det\left(K - \omega^2 M\right) = 0,$$

which is an nth degree polynomial for ω^2 with, in general, n roots corresponding to n different normal modes. Once you have solved this determinant equation for the frequencies ω_i you must go back to the full generalized eigenvalue matrix equation for the mode shapes $\mathbf{A}_i$. The full form for the general motion will then be the sum of these different normal modes:

$$\delta\mathbf{q} = \sum_{i=1}^{n} C_i \mathbf{A_i} \cos(\omega_i t + \phi_i),$$

where the constants C_i and ϕ_i in the motion will be set by the initial conditions. Therefore, the full behavior of the system will be a sum of n different vibrations, each with a particular shape and frequency.



9.3 Back to the mass and spring example

In this case, if we expand the Lagrangian

$$\mathcal{L}((x,y),(\dot{x},\dot{y})) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(x-l)^2 - \frac{1}{2}k(3l-y-l)^2 - \frac{1}{2}k(y-x-l)^2.$$

around the equilibrium position,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} l \\ 2l \end{pmatrix} + \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

we get

$$\mathcal{L} = \frac{1}{2}m\delta\dot{x}^2 + \frac{1}{2}m\delta\dot{y}^2 - \frac{1}{2}k\delta x^2 - \frac{1}{2}k\delta y^2 - \frac{1}{2}k(\delta x - \delta y)^2.$$

In this case, we didn't have to do any Taylor expanding to get a quadratic Lagrangian because springs actually have quadratic potential energies, but it won't always be so simple. We can immediately write the Lagrangian in matrix form as

$$\mathcal{L} = \frac{1}{2}(\delta\dot{x},\delta\dot{y})\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \end{pmatrix} - \frac{1}{2}(\delta x,\delta y)\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

and therefore read off the M and K matrices as:

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad K = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix},$$

which explains why they are called M and K.

We now know we are searching for two normal modes in the form $(\delta x,\delta y) = \mathbf{A}e^{i\omega t}$, which solve

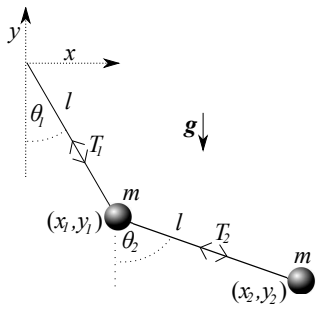
$$-\omega^2 M\mathbf{A} = -K\mathbf{A} \quad \Longrightarrow \quad -\omega^2\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} A_x \\ A_y \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} A_x \\ A_y \end{pmatrix}.$$

We can now either simply guess the mode shapes $(1,1)$ and $(-1,1)$ and evaluate the frequencies directly from this equation to get $\omega^2 = \frac{k}{m}, \frac{3k}{m}$, or we can use the determinant method to solve

$$\det\left(K - \omega^2 M\right) = 0 \quad \Longrightarrow (2k-\omega^2 m)^2 - k^2 = 0 \quad \Longrightarrow \omega^2 = \frac{k}{m}, \frac{3k}{m}$$

and then solve the upper equation for the mode shapes to get $(1,1)$ and $(-1,1)$. Either way, we get the same solution as we did before.

9.4 Normal modes of the double pendulum



Let us now consider, again, the double pendulum, which has Lagrangian:

$$\mathcal{L}(\theta_1,\theta_2,\dot\theta_1,\dot\theta_2)=\frac{1}{2}ml^2\left(2\dot\theta_1^2+\dot\theta_2^2+2\dot\theta_1\dot\theta_2\cos(\theta_2-\theta_1)\right)+mgl(2\cos\theta_1+\cos\theta_2).$$

The equilibrium position is as $\theta_1=\theta_2=0$, so we expand the Lagrangian to second order around this position, setting

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}+\begin{pmatrix}\delta\theta_1\\ \delta\theta_2\end{pmatrix}.$$

The expanded Lagrangian is:

$$\mathcal{L}((\delta\theta_1,\delta\theta_2),(\delta\dot\theta_1,\delta\dot\theta_2))=\frac{1}{2}ml^2\left(2\delta\dot\theta_1^2+\delta\dot\theta_2^2+2\delta\dot\theta_1\delta\dot\theta_2\right)+mgl\left(3-\delta\theta_1^2-\frac{\delta\theta_2^2}{2}\right),$$

which we can write in matrix form as

$$\mathcal{L}((\delta\theta_1,\delta\theta_2),(\delta\dot\theta_1,\delta\dot\theta_2))=\frac{1}{2}\left(\delta\dot\theta_1,\delta\dot\theta_2\right)\begin{pmatrix}2ml^2 & ml^2\\ ml^2 & ml^2\end{pmatrix}\begin{pmatrix}\delta\dot\theta_1\\ \delta\dot\theta_2\end{pmatrix}-\frac{1}{2}\left(\delta\theta_1,\delta\theta_2\right)\begin{pmatrix}2mgl & 0\\ 0 & mgl\end{pmatrix}\begin{pmatrix}\delta\theta_1\\ \delta\theta_2\end{pmatrix}.$$

To find the normal modes of oscillation, $(\delta\theta_1,\delta\theta_2)=(A_{\theta1},A_{\theta2})\mathrm{e}^{i\omega t}$we now need to solve $\det\left(K-\omega^2M\right)=0$. It is very helpful to cancel out a factor of ml^2 from, which gives us

$$\left|\begin{pmatrix}2g/l-2\omega^2 & -\omega^2\\ -\omega^2 & g/l-\omega^2\end{pmatrix}\right|=0\quad\rightarrow\quad\left(\frac{2g}{l}-2\omega^2\right)\left(\frac{g}{2}-\omega^2\right)-\omega^4=0\quad\rightarrow\quad\omega_\pm^2=\frac{g}{l}\left(2\pm\sqrt{2}\right).$$

To find the mode shapes, we return to

$$\omega^2M\begin{pmatrix}A_{\theta1}\\ A_{\theta2}\end{pmatrix}=K\begin{pmatrix}A_{\theta1}\\ A_{\theta2}\end{pmatrix}.$$

Substituting in the expected frequencies, and the M and K matrix,

$$\frac{g}{l}\left(2\pm\sqrt{2}\right)\begin{pmatrix}2ml^2 & ml^2\\ ml^2 & ml^2\end{pmatrix}\begin{pmatrix}A_{\theta1}\\ A_{\theta2}\end{pmatrix}=\begin{pmatrix}2mgl & 0\\ 0 & mgl\end{pmatrix}\begin{pmatrix}A_{\theta1}\\ A_{\theta2}\end{pmatrix},$$

we see the two solutions are

$$\begin{pmatrix}A_{\theta1\pm}\\ A_{\theta2\pm}\end{pmatrix}=\begin{pmatrix}-1\mp\sqrt{2}\\ 2\pm\sqrt{2}\end{pmatrix}\approx\begin{pmatrix}0.414214\\ 0.585786\end{pmatrix},\begin{pmatrix}2.41421\\ -3.41421\end{pmatrix}.$$

As we might have guessed, the two modes are a low frequency in-phase mode, where the two masses swing in the same sense, and a a high frequency out-of-phase mode where the two masses move in opposite directions. However, we needed to do the whole calculation to get the actual frequencies and mode shapes, they aren't as simple as $(1,1)$ and $(1-1)$ for the spring problem.

9.5 Zero frequency modes

On some occasions, one of the normal modes has $\omega^2 = 0$. This means that when we substituted $\delta q = \mathbf{A}e^{i\omega t}$ into the equation of motion:

$$M\delta\ddot{\mathbf{q}} = -K\delta\mathbf{q} \qquad \rightarrow \qquad -\omega^2 M\mathbf{A} = -K\mathbf{A}$$

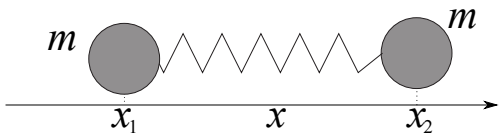
we found a mode shape with $K\mathbf{A} = 0$, and hence concluded $\omega^2 = 0$. In this case, we really have $\delta\ddot{\mathbf{q}} = 0$ so the correct solution is not oscillatory, but rather

$$\delta\mathbf{q} = (C_1 + C_2 t)\,\mathbf{A}.$$

Such a mode is known as a translational mode, and, stereotypically, involves the entire system moving through space at constant velocity.

9.5.1 Two masses connected by a spring

Consider two masses, m, in 1D and connected by a spring k. The Lagrangian is



$$\mathcal{L}(x_1, x_2, \dot{x}_1, \dot{x}_2) = \frac{1}{2}m\left(\dot{x_1}^2 + \dot{x_2}^2\right) - \frac{1}{2}k\left(x_2 - x_1 - l\right)^2.$$

There is an equilibrium state at $x_1 = 0, x_2 = l$, so we consider moving a small distance from this state $x_1 = 0 + \delta x_1$, $x_2 = l + \delta x_2$. Expanding the Lagrangian, we get

$$\mathcal{L}(\delta x_1, \delta x_2, \delta\dot{x}_1, \delta\dot{x}_2) = \frac{1}{2}m\left(\dot{\delta x_1}^2 + \dot{\delta x_2}^2\right) - \frac{1}{2}k\left(\delta x_2 - \delta x_1\right)^2,$$

from which we can read off

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \qquad K = \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}.$$

The generalized eigenvalue problem for the normal modes is

$$-\omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\mathbf{A} = -\begin{pmatrix} k & -k \\ -k & k \end{pmatrix}\mathbf{A}.$$

By inspection, one eigenvector is $\mathbf{A}_1 = (1, 1)$, which has $\omega^2 = 0$, and the second is $(1, -1)$, which has $\omega^2 = 2k/m$. Since one of these modes is a zero frequency "translational" mode, the general solution for free motion of the system is:

$$\begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix} = (C_1 + C_2 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_3\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos\left(\sqrt{\frac{2k}{m}}t + \phi\right).$$

9.6 "Orthogonality" of modes

You may have noticed that the mode shapes in the two-mass three-spring problem are $\mathbf{A}_1 = (1, 1)$ and $\mathbf{A}_2 = (1, -1)$, which are orthogonal (aka "normal") to each other, $\mathbf{A}_1.\mathbf{A}_2 = 0$, whereas in the double pendulum problem the modes were not orthogonal.

This leads us to wonder when normal modes are orthogonal. We recall that normal modes satisfy the generalized eigenvalue problem:

$$\omega^2 M\mathbf{A} = K\mathbf{A},$$

where M and K are symmetric matrices. In the two-mass three-spring problem, we actually had something more, which is that the mass matrix is a multiple of the identity matrix, $M = mI$. This turns the problem into:

$$K\mathbf{A} = m\omega^2\mathbf{A},$$

so $\mathbf{A}$ is a regular eigenvector of K. Since K is symmetric, its eigenvectors are orthogonal. A similar thing happens if K is a multiple of the identity matrix instead: then $\mathbf{A}$ is an eigenvector of the symmetric matrix M.

In such cases, realizing the modes are orthogonal is extremely useful, as it allows us to "spot" modes by looking for modes orthogonal to existing solutions. For example, having found $\mathbf{A}_1 = (1, 1)$, if we know the modes are orthogonal, we immediatly know the other mode is going to be $\mathbf{A}_2 = (1, -1)$. It is then very useful to represent the modes in normalized form, so that the length of each vector is one and the modes form an ortho-normal set of vectors. In the two-mass three-spring case, this would entail setting

$$\mathbf{A}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \mathbf{A}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

The usefulness comes when we want to fit the initial conditions of the motion. Suppose we know that the system starts at rest, but from position $(\delta x, \delta y) = (1, 3)$. Given we are starting at rest, the general form for the solution will be

$$\begin{pmatrix}\delta x\\\delta y\end{pmatrix} = C_1\mathbf{A}_1\cos(\omega_1 t) + C_2\mathbf{A}_2\cos(\omega_2 t),$$

so the initial position is simply

$$\begin{pmatrix}\delta x\\\delta y\end{pmatrix} = C_1\mathbf{A}_1 + C_2\mathbf{A}_2 = \begin{pmatrix}1\\3\end{pmatrix}.$$

This is equivalent to representing the vector $(1, 3)$ in the orthonormal basis $\mathbf{A}_1$ $\mathbf{A}_2$, so all we need to do take dot both sides with $\mathbf{A}_1$ and $\mathbf{A}_2$ to find $C_1 = \mathbf{A}_1.(1, 3)$, $C_2 = \mathbf{A}_2.(1, 3)$, and we have a full solution to the problem.

More generally, neither K nor M is a multiple of the identity matrix, the modes are not orthogonal, and fitting initial conditions is more work. For example, in the two-mass one-spring problem, we considered two different masses, so M was not a multiple of identity (and nor was K) and the modes are indeed not orthogonal. We might wonder whether there is any equivalence of orthogonality for the generalized eigenvalue problem. If you step through the normal proof of orthogonality, but with symmetric matrices on both sides of the equation, you will discover that, in general,

$$\mathbf{A}_i^T M\mathbf{A}_j = 0 \quad \text{and} \quad \mathbf{A}_i^T K\mathbf{A}_j = 0,$$

i.e. the modes are "orthogonal" if you put either the M or K matrix in the middle of the dot product. This is much less useful.

9.7 Normal coordinates (Non examinable)

Once we have found our normal modes, $\mathbf{A}_i$, and their frequencies ω_i, the general solution for free vibration of the system is:

$$\delta\mathbf{q} = \sum_i C_i \mathbf{A}_i \cos(\omega_i t + \phi_i),$$

where C_i and ϕ_i, the amplitude and phase of each normal mode, are determined by the initial conditions of the motion. In this solution we see that the individual modes have constant amplitude during the motion: the modes are independent, with no transfer of amplitude or energy from one mode to another.

A good plan is to change to new generalized coordinates Q_i, where each mode is captured by exactly one of the new coordinates. This means that, when Q_i is varied, the system moves in the ith mode, so we need to write:

$$\delta\mathbf{q} = \sum_i Q_i \mathbf{A}_i.$$

These are linear equations for q_i in terms of Q_i, but if we wish, we can also invert them to find $Q_i(\delta\mathbf{q})$. If we work with these new coordinates, the equation of motion for each Q_i, must be simply

$$\ddot{Q}_i = -\omega_i^2 Q_i,$$

the equation of motion for a single degree of freedom harmonic oscillator. It follows that if we make this coordinate change in our Lagrangian we must get the from

$$\mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}}) = \sum_i \frac{1}{2} m_i \left(\dot{Q}_i^2 - \omega_i^2 Q_i^2 \right)$$

with some set of mass values m_i. This, again, is simply the Lagrangian for a set of independent one degree of freedom harmonic oscillators. We have already seen an example of this: for the case of two masses connected by a spring, we originally wrote the Lagrangian in terms of the displacements of the masses, x_1 and x_2, but the shapes of the normal modes turned out to be $(1, 1)$ and $(1, -1)$, corresponding to $Q_1 = \frac{1}{2}(x_1 + x_2)$, which is the position of the center of mass, and $Q_2 = (x_1 - x_2)$ which is the separation of the masses. We have already seen that, if we write this Lagrangian in terms of the center of mass and the separation, it does indeed turn into two separate Lagrangians.

More generally, from this we see that the normal modes analysis actually boils down to finding a coordinate set Q_i for which our n degree of freedom system Lagrangian turns into a sum of n one degree of freedom Lagrangians. We have in fact, again, managed to reduce a complicated problem back into a (set of) one degree of freedom problems, we just had to do rather more work this time.

9.8 Unstable equilibria

In general, there is no guarantee that the equilibrium we are expanding around will be a minimum in the potential energy. If it is a maximum, we will find that the ω^2 values we get are negative, implying that $\omega = \pm i/\tau$ is imaginary.

In this case, as in the one degree of freedom case, the solutions change from being vibrations to being exponentially growing and decaying with time constant τ:

$$\mathbf{A}\mathrm{e}^{i\omega t} \to \mathbf{A}\mathrm{e}^{\pm t/\tau}.$$

This corresponds to the equilibrium solutions being unstable as, given a small perturbation, the perturbation will grow exponentially over the time scale τ. As in the one d.o.f. case, it is useful to calculate τ to get a sense of how long you have before a small perturbation becomes a very big one.

In general, you will be at a saddle point in potential energy, where some ω will be real and others imaginary. Such a state is still unstable, as any small perturbation involving the unstable mode will then grow exponentially. This is much like a ball resting at an actual saddle point in a mountain range: it is certainly unstable to rolling off in the downhill direction.

10 Using a computer

Having found the equilibrium points and the normal-modes/growth-rates near these equilibria we have learnt a lot about motion of the double pendulum. To go further, we must resort to a computer. In this section, we will discuss some strategies for numerically integrating dynamics equations, and then deploy them to study the motion of the double pendulum.

Lagrangian mechanics produces second order equations of motion which take the form:

$$\ddot{\mathbf{q}} = \mathbf{f}(\dot{\mathbf{q}}, \mathbf{q}).$$

For example, for the mass-string double pendulum, we can write the equations of motion as:

$$\ddot{\theta}_1 = \frac{g(\sin(\theta_1 - 2\theta_2) + 3\sin(\theta_1)) + 2l\sin(\theta_1 - \theta_2)\left(\dot{\theta}_1^2\cos(\theta_1 - \theta_2) + \dot{\theta}_2^2\right)}{l(\cos(2(\theta_1 - \theta_2)) - 3)}$$

$$\ddot{\theta}_2 = -\frac{2\sin(\theta_1 - \theta_2)\left(2g\cos(\theta_1) + 2l\dot{\theta}_1^2 + l\dot{\theta}_2^2\cos(\theta_1 - \theta_2)\right)}{l(\cos(2(\theta_1 - \theta_2)) - 3)}.$$

For clarity we will initially focus on a 1-D particle moving with the traditional Newtonian equation of motion:

$$\ddot{x} = \frac{F(x, \dot{x}, t)}{m},$$

which, for the important case of motion in a potential energy, simplifies to

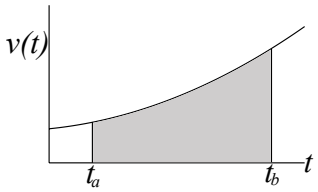
$$\ddot{x} = -\frac{1}{m}F(x) = -\frac{1}{m}V'(x).$$

However, as we will see at the end of the section, the same approaches also work for integrating general Lagrangian systems such as the double pendulum.

The detailed integration algorithms provided in this section **are for reference only**. In the exam, you might be asked to describe a forward Euler integrator for a Lagrangian system, or to show an awareness of errors, orders, stability, and chaos, but you will not be asked to reproduce the more complicated integration schemes from memory.

10.1 Computing integrals

Before we consider integrating equations of motion, we recall how to calculate a regular "*quadrature*" integral numerically, as we might need to do for the second-integral in one d.o.f. dynamics. Suppose we know the function $v(t)$ and wish to calculate

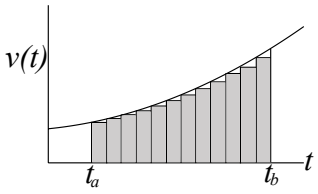


$$x = \int_{t_a}^{t_b} v(t)\mathrm{d}t.$$

The integral is equivalent to the area under a curve. To calculate the integral, we first break it into much N small time intervals, each of duration $\Delta t = (t_b - t_a)/N$. Defining the set of boundary times $t_{n+1} = t_n + \Delta t$, the integral becomes a sum of little integrals:

$$x = \int_{t_a}^{t_b} v(t)\mathrm{d}t = \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} v(t)\mathrm{d}t.$$

We next need to calculate a value for each of these little areas/integrals. The simplest approximation replaces each little area with a rectangle. We might take the height of the rectangle as the height of the function at the left hand side:



Left Riemann sum.

$$\int_{t_n}^{t_{n+1}} v(t)\mathrm{d}t \approx v(t_n)\Delta t,$$

which is an approximation known as the "left Riemann sum." Intuitively, if we make Δt small enough, this sum will produce a very good approximation for the real integral. To make this notion more precise, we can estimate the error we make in approximating each little rectangle by using a Taylor expansion in t around t_n:

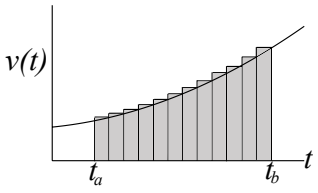
$$v(t) = v(t_n) + (t - t_n)v'(t_n) + ...$$

If we now do the integral

$$\int_{t_n}^{t_{n+1}} v(t)\mathrm{d}t = \int_{t_n}^{t_{n+1}} v(t_n) + (t - t_n)v'(t_n) + ...\mathrm{d}t = v(t_n)\Delta t + \frac{1}{2}v'(t_n)\Delta t^2 + ...$$

we can see that we are throwing away a term of size $\mathcal{O}(\Delta t^2)$ in each rectangle approximation, which corresponds to the little white triangle between each rectangle and the true curve. However, the whole integral contains $N = (t_b - t_a)/\Delta t$ rectangles, so the total error will be of size $N\Delta t^2 \sim \Delta t$. This approximation is called a "first order" integration method: if we want to reduce the error in our answer by a factor of 100, we need to reduce Δt by a factor of 100, and therefore use 100 times more rectangles in the sum. In principle, if we make Δt small enough, we can get as close as we like to the right answer. In practice, computers store numbers with a finite number of significant digits, so the answer will eventually stop improving once the computer can't resolve the change in function value from one rectangle to the next.

We could also use a "right Riemann sum" method:



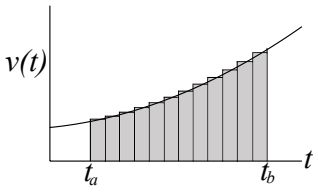
Right Riemann sum.

$$\int_{t_n}^{t_{n+1}} v(t)\mathrm{d}t \approx v(t_{n+1})\Delta t$$

which is also a first order method.

Alternatively, we could be much smarter, and use the function value at the mid-point as the height of the rectangle:

$$\int_{t_n}^{t_{n+1}} v(t)\mathrm{d}t \approx v(t_n + \Delta t/2)\Delta t.$$



Mid point method.

From the diagram, we can see the mid-point method gives a much better approximation. Again, we can examine the error by using a Taylor series, this time around the mid-point $t_n + \Delta t/2$:

$$\begin{aligned}\int_{t_n}^{t_{n+1}} v(t)\mathrm{d}t &= \int_{t_n}^{t_{n+1}} v(t_n + \Delta t/2) + v'(t_n + \Delta t/2)(t - (t_n + \Delta t/2)) + \mathcal{O}(\Delta t^2)\mathrm{d}t \\ &= v(t_n + \Delta t/2)\Delta t + \mathcal{O}(\Delta t^3),\end{aligned}$$

so the error on each rectangle is $\mathcal{O}(\Delta t^3)$, and the error on the whole integral is $N\mathcal{O}(\Delta t^3) \sim \mathcal{O}(\Delta t^2)$. We call this a second order method. To reduce error by a factor of 100, we only need ten times as many points in the sum.

In practice, you should never use the forward or backwards rectangle methods, as the mid-point method is no more work and is much more accurate. The forward and backward methods only produce exactly the right answer if the function is a constant, whereas the mid-point method is also exact if for all straight lines.

Other methods also exist. A close relative of the mid-point method is the trapezoidal method, where each integral is approximated by a trapezium based on the function values at t_n and t_{n+1}. This method is also second order, but tends to produce bigger errors. An even higher order method is Simpson's method, which uses the function value at the start, middle and end of the range to make the approximation

$$\int_{t_n}^{t_{n+1}} v(t)\mathrm{d}t \approx \frac{1}{6}\left(v(t_n) + 4v\left(\left(t_n + t_{n+1}\right)/2\right) + v(t_{n+1})\right)\Delta t.$$

This is clearly more work, but in return we get an answer up to $\mathcal{O}(\Delta t^5)$ for each mini integral, and hence a total error $\mathcal{O}(\Delta t^4)$ for the whole integral. To make the answer 100 times more accurate, we only need $100^{1/4} \approx 3$ times as many points.

10.2 Euler Method

10.2.1 Forward Euler

In mechanics we need to integrate a differential equation for the motion, rather than just computing a standard area integral. A simpler intermediate problem would be to find $y(t)$ that satisfies the first order differential equation:

$$\frac{dy}{dt} = f(y,t).$$

If $f(y,t) = f(t)$ doesn't depend on y then this would be a simple case of doing an integral. Applying the "left Riemann sum" rectangle method would give

$$y(t_{n+1}) \equiv y(t_n + \Delta t) \approx y(t_n) + f(t_n)\Delta t.$$

The forward Euler method for solving first order differential equations simply applies this same approach, but allowing y dependence in $f(y,t)$

$$y(t_{n+1}) = y(t_n) + f(y(t_n), t_n)\Delta t.$$

This approach is very intuitive: we step y forward in time from $y(t_n)$ to $y(t_{n+1})$ by assuming it moves at a rate evaluated at t_n. If Δt is small enough, this the rate will hardly change between t_n and t_{n+1}, so this will be a very good approximation. We can analyze the error in this method by Taylor expanding y around t_n:

$$y(t_{n+1}) = y(t_n) + f(y(t_n), t_n)\Delta t + \mathcal{O}\left(\Delta t^2\right),$$

which shows that the method commits a $\mathcal{O}(\Delta t^2)$ error each time step, and hence, like the "right Riemann sum," is a first order method producing an $\mathcal{O}(\Delta t)$ error over the whole motion.

To use this method in mechanics, there is an immediate problem. Our equations of motion, like $F = ma$, are second order differential equations, not first order. The trick is to treat the velocity itself as a second variable, so the second order equation

$$\frac{d^2x}{dt^2} = \frac{F(x, \dot{x}, t)}{m},$$

becomes two first order equations,

$$\begin{aligned} \dot{x} &= v(t) \\ \dot{v} &= \frac{F(x, v, t)}{m} \equiv a(x, v, t). \end{aligned}$$

We can then apply the forward Euler method to both equations at once, leading to the update rule:

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + v(t_n)\Delta t \\ v(t_{n+1}) &= v(t_n) + a(x(t_n), v(t_n), t_n)\Delta t. \end{aligned}$$

The forward Euler method has the advantage that it is intuitive, easy to code, and can be used on any equation of motion. However, there are two big problems. Firstly, it is only an $\mathcal{O}(\Delta t)$ method, so you need a very small Δt to be accurate. The second problem is more subtle: when you make a small error in position, you end up in the wrong location, and therefore calculate the wrong force. This wrong force makes the subsequent positions even more wrong, and the forces even more wrong. For many mechanical systems, this problem makes the forward Euler method horribly unstable and, over time, the system often goes haywire. This kind of problem is the key difference between integrating a differential equations, $\frac{dy}{dt} = f(y, t)$ and computing a simple integral like $\frac{dy}{dt} = f(t)$, and makes the forward Euler method a poor choice for most mechanics problems. Intuitivley, the particle has a tendency to overshoot its real position, and hence get to higher force/potential energy regions, which then causes the particle to overshoot even more, and over time the energy/velocity/position start to diverge.

10.2.2 Backward Euler or Implicit Euler

If the forward Euler is the equivalent of the right Riemann sum, backward Euler is the equivalent of the left Riemann sum. It works in the same way, but approximating the acceleration and velocity at the end of each time step rather than the begining

$$y(t_{n+1}) = y(t_n) + f(y(t_{n+1}), t_{n+1})\Delta t.$$

This is still committing a first order method. However, now the updated positions $y(t_{n+1})$ appear on both sides of the above equation, so rather than simply computing $y(t_{n+1})$, we have to solve for $y(t_{n+1})$ numerically. For this reason, this is called the implicit Euler method. This method is no more accurate than forwards Euler, and is much more work for both the programmer and the computer. However, it turns out to be far more numerically stable than forward Euler, so it is occasionally used when numerical stability is a problem.

For a dynamics problem, again writing $F = ma$ as two first order equations, we have

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + v(t_{n+1})\Delta t \\ v(t_{n+1}) &= v(t_n) + a(x(t_{n+1}), v(t_{n+1}), t_{n+1})\Delta t, \end{aligned}$$

which have to be solved, at each time step, for $x(t_{n+1})$ and $v(t_{n+1})$.

10.2.3 Semi-implicit Euler (Non examinable)

The semi-implicit Euler method (also known as symplectic Euler, semi-explicit Euler and Euler-Cromer) is half way between the forward and backwards methods: you update the velocities in a forward Euler way, but then update the positions using the updated velocities, i.e. in a backward Euler way:

$$v(t_{n+1}) = v(t_n) + a(x(t_n), v(t_n), t_n)\Delta t$$
$$x(t_{n+1}) = x(t_n) + v(t_{n+1})\Delta t.$$

Although this is called "Semi-implicit," in fact these equations are explicit equations for $v(t_{n+1})$ and $x(t_{n+1})$ so no solving is required.

You can also do this the other way round, and first update the positions in a forward way then the velocities in a backwards way. In general this does lead to implicit equations for v, but if the forces are not velocity dependent one gets simple explicit equations:

$$x(t_{n+1}) = x(t_n) + v(t_n)\Delta t$$
$$v(t_{n+1}) = v(t_n) + a(x(t_{n+1}), t_{n+1})\Delta t.$$

The semi-implicit Euler method is no harder to code or run than forward Euler, and is much more stable. In particular, semi-implicit Euler conserves energy (almost) perfectly, which prevents solutions from diverging wildly from the correct trajectory. Even though it is a first order method, this energy conserving property tends to keep the solutions looking reasonable (even if they aren't terribly accurate) and for this reason, semi-implicit Euler is the standard choice in physics engines for games and graphics.

10.3 Second order methods

However, in engineering, we often need precise solutions, not just solutions that aren't obviously nonsense. This motivates us to look for an equivalent of the mid-point method, where the error scales as $\mathcal{O}(\Delta t^2)$. There are three almost identical mid-point like algorithms that are used for this task. They are all restricted to dynamics problems where the forces depend only on position:

$$\dot{x} = v(t)$$
$$\dot{v} = \frac{F(x,t)}{m} = a(x,t).$$

10.3.1 Leapfrog

The leapfrog method works by interleaving our time points t_n with mid-point times $t_{n+1/2} = t_n + \Delta t/2$. The idea is then to evaluate velocities at the mid-points, and positions at the integer points:

$$x(t_{n+1}) = x(t_n) + v(t_{n+1/2})\Delta t$$
$$v(t_{n+3/2}) = v(t_{n+1/2}) + a(x(t_{n+1}), t_{n+1})\Delta t.$$

By doing this, the position and velocity are both updated by a whole time step based on their rate of change at the mid-point of their time increment. The error in the method can be analyzed in the exact same way as the regular the mid-point method, i.e. by Taylor expanding around the mid-point:

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + \int_{t_n}^{t_{n+1}} v \, \mathrm{d}t \\ &= x(t_n) + \int_{t_n}^{t_{n+1}} v(t_{n+1/2}) + v'(t_{n+1/2})(t - t_{n+1/2}) + \frac{1}{2}v''(t_{n+1/2})(t - t_{n+1/2})^2 + \ldots \, \mathrm{d}t \\ &= x(t_n) + v(t_{n+1/2})\Delta t + \mathcal{O}(\Delta t^3) \end{aligned}$$

and similarly for $v(t)$. The leapfrog approach commits a $\mathcal{O}(\Delta t^3)$ error every time step, and hence accumulates a total error $\mathcal{O}(\Delta t^2)$.

10.3.2 Velocity-verlet (Non examinable)

The leapfrog method is a little inconvenient, as it doesn't tell you the position and velocity at the same time point. To get round this, one can break the velocity update rule into two steps, one which updates from $v(t_{n+1/2})$ to $v(t_{n+1})$, and second which updates from $v(t_{n+1})$ to $v(t_{n+3/2})$.

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + v(t_{n+1/2})\Delta t \\ v(t_{n+1}) &= v(t_{n+1/2}) + a(x(t_{n+1}), t_{n+1})\Delta t, \\ v(t_{n+3/2}) &= v(t_{n+1}) + a(x(t_{n+1}), t_{n+1})\Delta t. \end{aligned}$$

This looks confusing, but if you think the step immediately prior was

$$v(t_{n+1/2}) = v(t_n) + \frac{1}{2}a(x(t_n), t_n)\Delta t$$

and you can then substitute this expression for $v(t_{n+1/2})$ into the two two stages above to eliminate the appearance of half steps altogether:

$$\begin{aligned} x(t_{n+1}) &= x(t_n) + v(t_n)\Delta t + \frac{1}{2}a(x(t_n), t_n)\Delta t^2 \\ v(t_{n+1}) &= v(t_n) + \frac{1}{2}\left(a(x(t_n), t_n) + a(x(t_{n+1}), t_{n+1})\right)\Delta t, \end{aligned}$$

which is known as the velocity-Verlet algorithm.

This algorithm looks sensible and intuitive: we have a trapezium like rule for updating velocities, and a second order Taylor series rule for updating position. Deriving it this way makes clear it is, in fact, equivalent to the Leapfrog algorithm, so it is also a $\mathcal{O}(\Delta t^2)$ method.

10.3.3 Traditional Verlet (Non examinable)

If we Taylor expand the position around t_n to estimate $x(t_{n+1})$ and $x(t_{n-1})$ we get:

$$x(t_{n+1}) = x(t_n) + v(t_n)\Delta t + \frac{1}{2}a(t_n)\Delta t^2 + \frac{1}{6}\dot{a}(t_n)\Delta t^3 + \mathcal{O}(\Delta t^4)$$
$$x(t_{n-1}) = x(t_n) - v(t_n)\Delta t + \frac{1}{2}a(t_n)\Delta t^2 - \frac{1}{6}\dot{a}(t_n)\Delta t^3 + \mathcal{O}(\Delta t^4).$$

Adding these two results together and taking $x(t_{n-1})$ onto the left hand side gives the traditional Verlet update rule for position:

$$x(t_{n+1}) = 2x(t_n) - x(t_{n-1}) + a(x(t_n), t_n)\Delta t^2,$$

which should be familiar from the Mars lander project. This argument leads to the (wrong) impression that traditional Verlet is a third order method, but actually it is exactly equivilent to leapfrog and velocity-Verlet, so it is a second order method. (Showing this is left as an exercise for the reader: consider three consecutive leapfrog steps, $x(t_n)$, $v(t_{n+1/2})$, $x(t_{n+1})$, and algebraically eliminate the velocities.) Our (subtle) mistake is that we are now effectively integrating a second order differential equation in one go, which requires two integrations (the clue is in the two in front of $x(t_n)$), and increases the accumulated error by two orders rather than one, giving a final error $\mathcal{O}(\Delta t^2)$.

Traditional Verlet is the oldest of this family of integrators, and rarely used in practice because it is generally useful to know velocities!

10.3.4 Stability of Verlet like methods (Non examinable)

The three Verlet-like methods are equivalent, and commonly used in practical dynamics integrators

In addition to being second order, they all share the important property of being very stable (unlike forward Euler), as they (almost exactly) conserve energy, momentum, angular momentum and other conserved quantities, which stops the system diverging into unphysical behaviour.

A key property of motion in a potential is that it is time-reversible. If the trajectory $x(t)$ is a solution, then so is $x(-t)$ in which the particle reverses along the same path. The stability of Verlet methods comes from the fact that they exactly respect this time reversibility: if, when you integrate forward in time you get a sequence of positions and velocities $x(t_0)...x(t_i)...x(t_N)$, $v(t_0)...v(t_i)...v(t_N)$ then, if you start from $x(t_N)$ $-v(t_N)$ and run the same integrator, you will get back the exact same set of points, finishing at $x(t_0)$, $-v(t_0)$. This exact property is because, in each time step, the update uses rate of change data at the mid-point, which is symmetric with regards to the two ends of the time step, so going forwards is the same as going backwards. We can show this symmetry exactly if we rearrange leapfrog method to tell us how to run backwards:

$$x(t_n) = x(t_{n+1}) - v(t_{n+1/2})\Delta t$$
$$v(t_{n+1/2}) = v(t_{n+3/2}) - a(x(t_{n+1}), t_{n+1})\Delta t,$$

which is exactly the original leapfrog forward rule but with negative velocity.

This reversibility/symmetry in time makes the algorithms very stable. If numerical instabilities such as increasing energy appear in the forward integration, they must also appear when integrating backwards, since forward and backward integrations are the same. However, since the backward integration gets back to the original starting position, the only possibility is that neither are unstable.

Numerical schemes that conserve quantities correctly are called symplectic. As the name suggest semi-implicit Euler (aka symplectic Euler) is another example, but this time the time reversibility is more subtle: if you run dynamics forwards with one flavour, you can run back exactly over the same points with the other flavour.

10.4 Fourth order Runge Kutta (Non examinable)

Just as forward Euler is the equivalent of the right Riemann sum ($\mathcal{O}(\Delta t)$), and Verlet integration is the equivalent of midpoint integration ($\mathcal{O}(\Delta t^2)$) there is also an equivalent of Simpson's rule ($\mathcal{O}(\Delta t^4)$. This method is known as fourth-order Runge Kutta, RK4 or, often, just Runge Kutta, and is the default choice for dynamics integration in science and engineering if high accuracy is required. Like forward Euler method, RK4 is for first order differential equations,

$$\dot{y} = f(t, y).$$

and integrates them by applying the update rule

$$y(t_{n+1}) = y(t_n) + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)\Delta t + \mathcal{O}(\Delta t^5),$$

where the k_i are estimates for the $\dot{y}$ at various points in the interval between t_n and t_{n+1}. More precisely, they are calculated as

$$\begin{aligned}
k_1 &= f(t_n, y(t_n)) \\
k_2 &= f(t_n + \Delta t/2, y(t_n) + k_1 \Delta t/2) \\
k_3 &= f(t_n + \Delta t/2, y(t_n) + k_2 \Delta t/2) \\
k_4 &= f(t_n + \Delta t, y(t_n) + k_3 \Delta t).
\end{aligned}$$

Intuitively, k_1 is the value of $\dot{y}$ at the start of the time step, k_2 is the value at the mid-point (with the value of y estimated using forward Euler and k_1) k_3 is also the value at the mid point (with y estimated using backward Euler and k_2) and k_4 is the value at the end of the time step, using y estimated with a mid-point rule and k_3. If f is independent of y, this reduces to evaluating the integrand at the start (k_1) mid-point (k_2 and k_3) and end (k_4) of the integration range, and exactly reproduces Simpson's rule.

To solve dynamics problem, we again $F = ma$ as two first order equations.

$$\begin{aligned}
\dot{x} &= v(t) \\
\dot{v} &= a(x, v, t),
\end{aligned}$$

and apply RK4 to them individually. More generally, in a multi degree of freedom Lagrangian dynamics problem, we will have a whole vector of first order differential equations rather than just two,

$$\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y}),$$

which we solve by applying RK4 to each component equation individually. To do this, we simply follow exactly the RK4 method above, except now $\mathbf{k}_i$ are also vectors. This general approach is a much more compact way of writing (and coding) the algorithm than writing out separate update rules for x and v as we did in the forward Euler section. As the name suggests, RK4 is a member of a whole family of Runge-Kutta integrators with different levels of complexity, and different orders or errors, but it is RK4 that is ubiquitous in practice.

10.5 Integration of the double pendulum (non examinable)

10.5.1 Implementation

In this section, as on the question sheet, we will consider a double pendulum governed by the generic double pendulum Lagrangian

$$\mathcal{L}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \frac{1}{2}\left(A\dot{\theta}_1^2 + B\dot{\theta}_2^2 + 2C\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right) + D\cos\theta_1 + E\cos\theta_2,$$

where the values of $A, ..., E$ depend on the details of double pendulum in question. As you showed examples paper 3, the Lagrangian equations of motion for θ_1 and θ_2 are

$$\begin{aligned} A\ddot{\theta}_1 + C\ddot{\theta}_2\cos(\theta_1 - \theta_2) + C\dot{\theta_2}^2\sin(\theta_2 - \theta_1) &= -D\sin\theta_1, \\ B\ddot{\theta}_2 + C\ddot{\theta}_1\cos(\theta_1 - \theta_2) - C\dot{\theta_1}^2\sin(\theta_2 - \theta_1) &= -E\sin\theta_2, \end{aligned}$$

which we can rearrange to make the accelerations the subjects, giving

$$\begin{aligned} \ddot{\theta}_1 =& \frac{-C\sin(\theta_1 - \theta_2)\left(B\dot{\theta}_2^2 + C\dot{\theta}_1^2\cos(\theta_1 - \theta_2)\right) - BD\sin(\theta_1) + CE\sin(\theta_2)\cos(\theta_1 - \theta_2)}{AB - C^2\cos^2(\theta_1 - \theta_2)} \\ \ddot{\theta}_2 =& \frac{C\sin(\theta_1 - \theta_2)\left(A\dot{\theta}_1^2 + C\dot{\theta}_2^2\cos(\theta_1 - \theta_2)\right) - AE\sin(\theta_2) + CD\sin(\theta_1)\cos(\theta_1 - \theta_2)}{AB - C^2\cos^2(\theta_1 - \theta_2)}. \end{aligned}$$

RK4 is the natural choice to numerically integrate these equations of motion, as we want accurate results (hence a high order method) and the equations clearly involve both acceleration and velocity. Our first task is to write these two second order differential equations as a vector of four first order equations. Using the notation of the previous section, we introduce the vector $\mathbf{y} = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ which stores the state of the system. The rate of change of the system state, $\dot{\mathbf{y}} = (\dot{\theta}_1, \dot{\theta}_2, \ddot{\theta}_1, \ddot{\theta}_2)$ is then given by:

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) = \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \frac{-C\sin(\theta_1-\theta_2)\left(B\dot{\theta}_2^2+C\dot{\theta}_1^2\cos(\theta_1-\theta_2)\right)-BD\sin(\theta_1)+CE\sin(\theta_2)\cos(\theta_1-\theta_2)}{AB-C^2\cos^2(\theta_1-\theta_2)} \\ \frac{C\sin(\theta_1-\theta_2)\left(A\dot{\theta}_1^2+C\dot{\theta}_2^2\cos(\theta_1-\theta_2)\right)-AE\sin(\theta_2)+CD\sin(\theta_1)\cos(\theta_1-\theta_2)}{AB-C^2\cos^2(\theta_1-\theta_2)} \end{pmatrix},$$

which are four first order equations, written in the form for using RK4 integration. The first step in writing an RK4 integrator is to program this function $\mathbf{f}$ into the computer, so that the computer can evaluate $\dot{\mathbf{y}}$ from a given value of $\mathbf{y}$. Then, at a given moment t_n during the integration, the computer is storing the current system state, $\mathbf{y}(t_n)$. To calculate the state at t_{n+1}, the computer must first calculate the four $\mathbf{k}_i$

$$\begin{aligned}\mathbf{k_1} &= \mathbf{f}(\mathbf{y}(t_n))\\ \mathbf{k_2} &= \mathbf{f}(\mathbf{y}(t_n) + \mathbf{k_1}\Delta t/2)\\ \mathbf{k_3} &= \mathbf{f}(\mathbf{y}(t_n) + \mathbf{k_2}\Delta t/2)\\ \mathbf{k_4} &= \mathbf{f}(\mathbf{y}(t_n) + \mathbf{k_3}\Delta t),\end{aligned}$$

and then apply the RK4 integration rule to compute $\mathbf{y}(t_{n+1})$.

$$\mathbf{y}(t_{n+1}) = \mathbf{y}(t_n) + \frac{1}{6}\left(\mathbf{k_1} + 2\mathbf{k_2} + 2\mathbf{k_3} + \mathbf{k_4}\right)\Delta t.$$

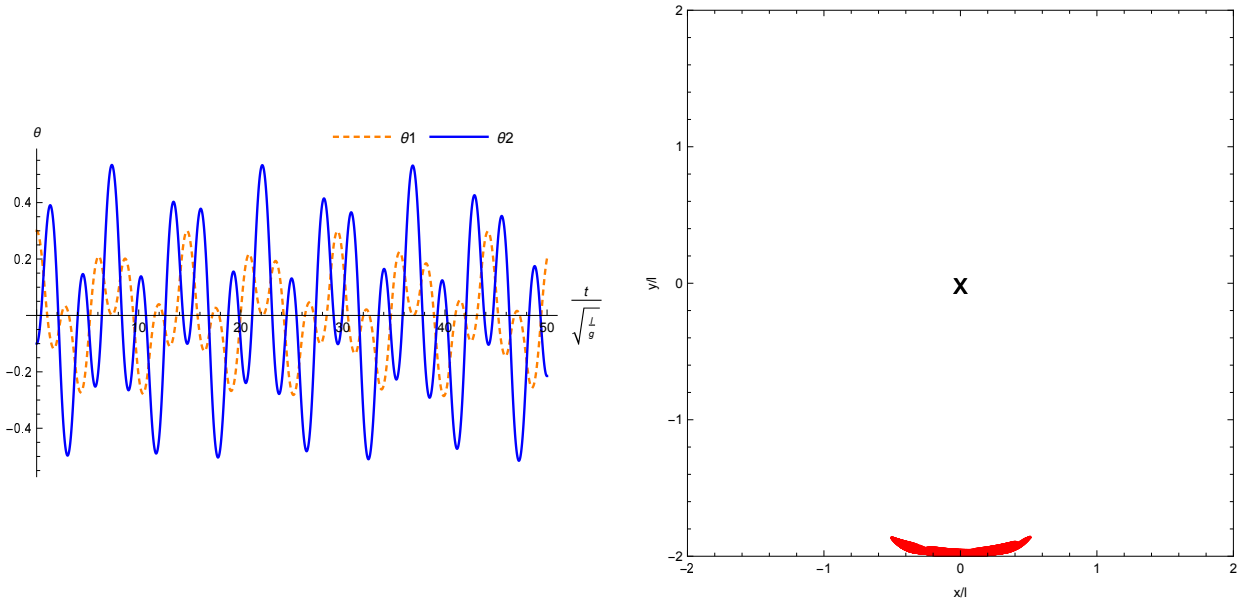
The initial conditions for the integration are supplied by the initial choice of $\mathbf{y}(0) = (\theta_1(0), \theta_2(0), \dot{\theta}_1(0), \dot{\theta}_2(0))$, which encodes the initial position and velocity of both rods of the double pendulum.

An RK4 integrator for the double pendulum, written in python, is supplied as part of Q6 on examples sheet 4. In the code, the vector $\mathbf{y}$ is referred to as *state*, the function $\mathbf{f}$ is implemented as *accels*, the $\mathbf{k}$ points are calculated and assembled in the function *RK4_grad*, and the actual updating of *state* is done in the function *integrate*. Using this code, as long as we choose a small enough value of Δt, we can integrate forward and calculate the motion of the double pendulum

10.5.2 Initial results

On the question sheet, you will integrate the equations of motion with $A, ..., E$ chosen to represent the lab double pendulum. In this section we will focus on a double pendulum consisting only of two uniform rigid rods of length l and mass m (for which, as you showed on examples paper 3, the values of $A, ..., E$ are $A = 4ml^2/3,\ B = ml^2/3,\ C = ml^2/2\ , D = 3mgl/2$ and $E = mgl/2$). The equations of motion only depend on m, g and l via the ratio g/l, which we set to 1, meaning we are measuring time in units of $\sqrt{l/g}$.

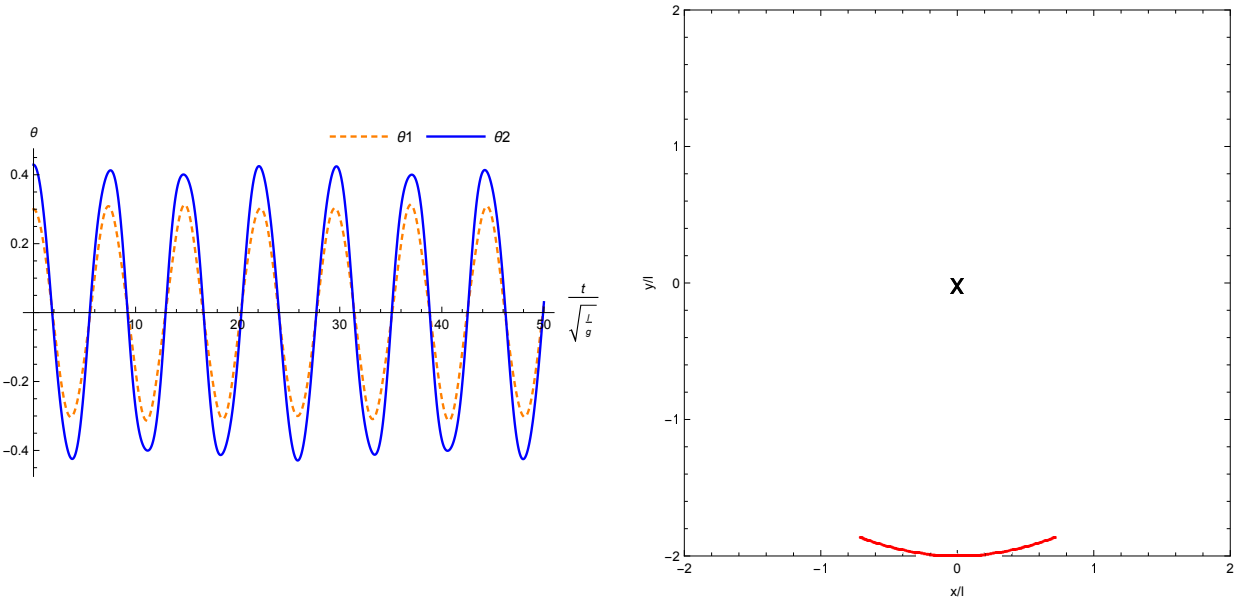
We start by integrating the equations of motion from a range of initial conditions of the form $(\theta_1(0), \theta_2(0), \dot{\theta}_1(0), \dot{\theta}_2(0)) = (\theta_1(0), \theta_2(0), 0, 0)$, corresponding to releasing the system from rest. The first motion we integrate starts from small angles, $\theta_1(0) = 0.3\text{rad}$ $\theta_2(0) = -0.1\text{rad}$. To visualize the result, we make a pair of plots: the left hand plot show θ_1 and θ_2 as a function of time, and the right hand plot shows the position traced out by the end of the pendulum.



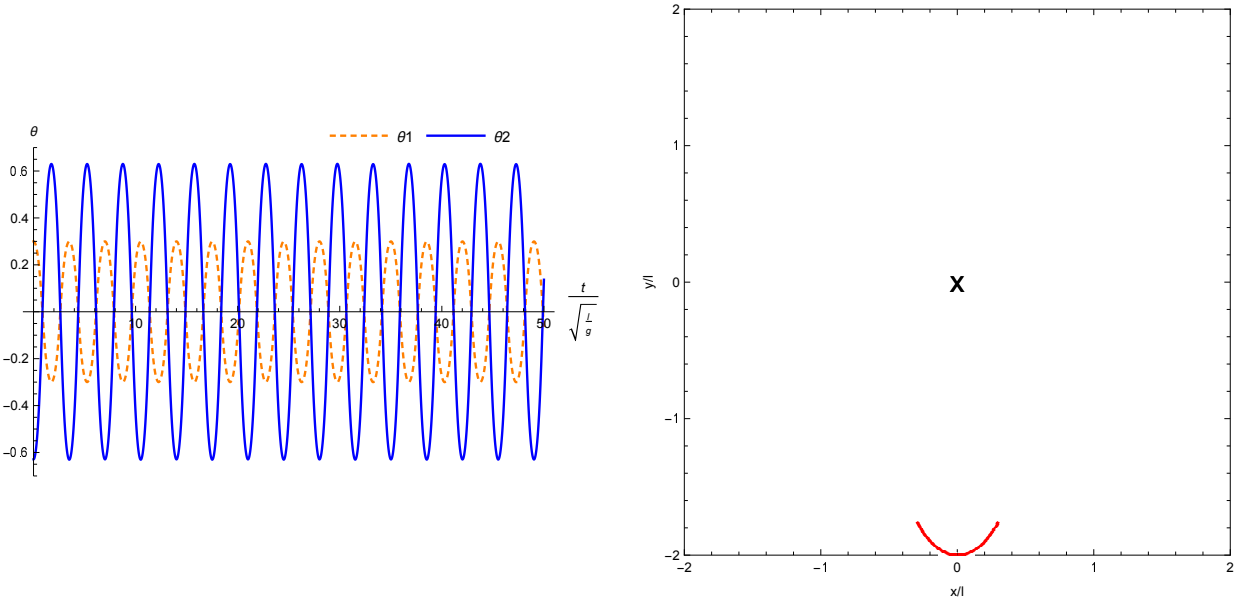
The above motion is complicated and difficult to interpret. However, we know that such low-amplitude motion follows a small-vibration "normal mode" solution which, in the case of the rigid-rod double pendulum, takes the form:

$$\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = C_1 \begin{pmatrix}0.57...\\ 0.819...\end{pmatrix} \cos\left(0.85...\sqrt{\frac{g}{l}}t + \phi_1\right) + C_2 \begin{pmatrix}0.43...\\ -0.90...\end{pmatrix} \cos\left(2.29...\sqrt{\frac{g}{l}}t + \phi_2\right).$$

The above plot is a mixture of both modes (i.e. both C_1 and C_2 are non zero) which makes it look confusing, but it is in fact an almost perfect match to the theoretical small amplitude motion. If we choose our initial conditions more carefully, we can see motions with only one normal mode active, which then produce a simple sinusoidal motion.

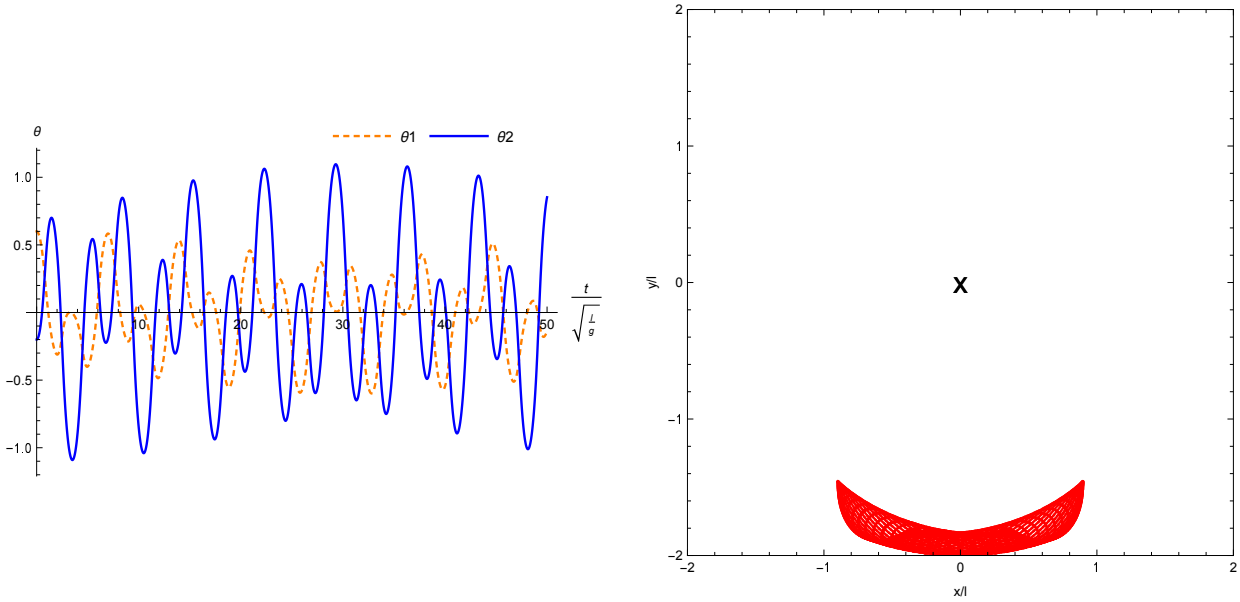


$\theta_1(0) = 0.300,\ \theta_2(0) = 0.429$: In phase normal mode with lower frequency.

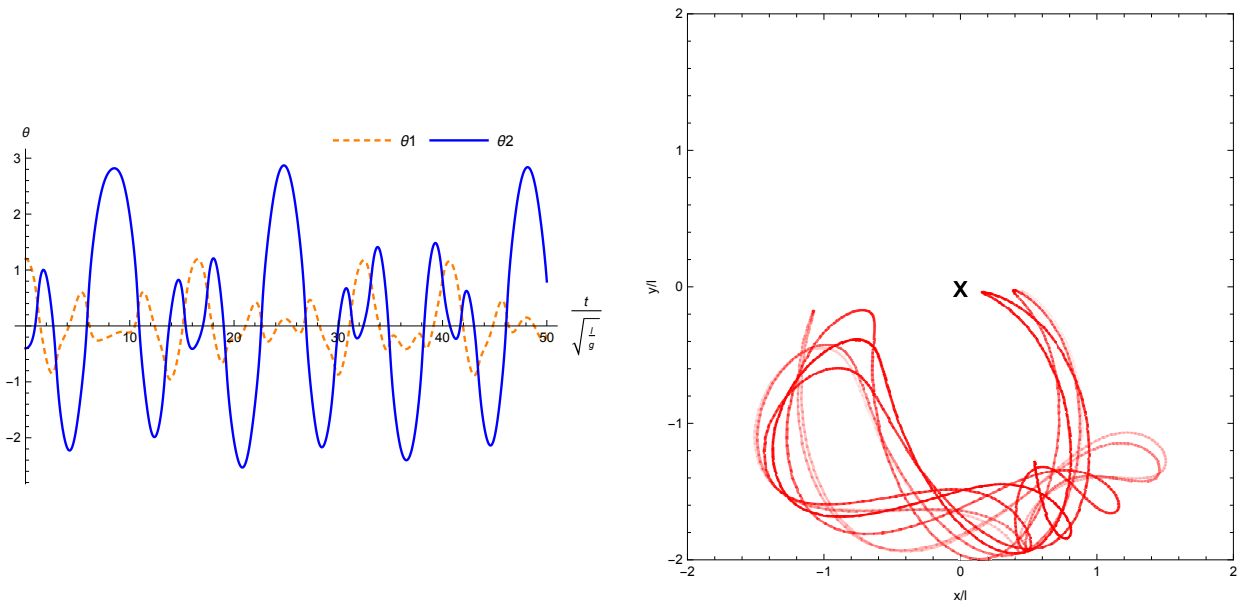


$\theta_1(0) = 0.300,\ \theta_2(0) = -0.629$: Out of phase normal mode with higher frequency.

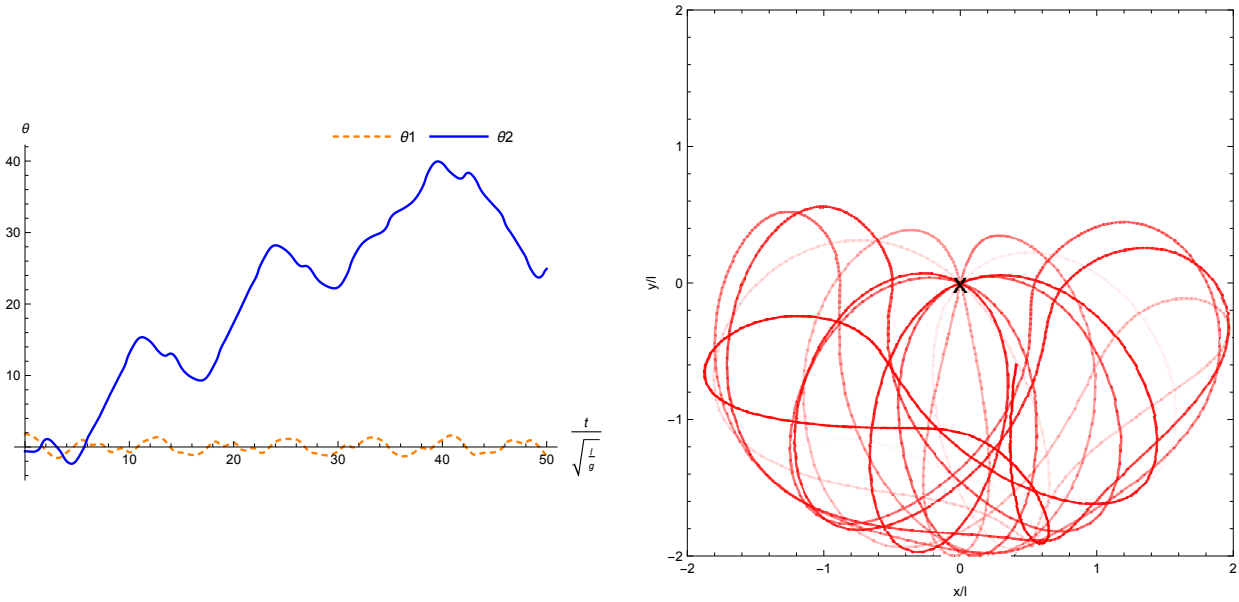
However, as we increase the initial angles further, and move out of the linearized regime, the motion becomes extremely complicated, especially once there is enough energy for the second pendulum to flip. The motion doesn't appear to be periodic or regular at all: the double pendulum keeps swinging in similar sorts-of ways, but doesn't strictly repeat.



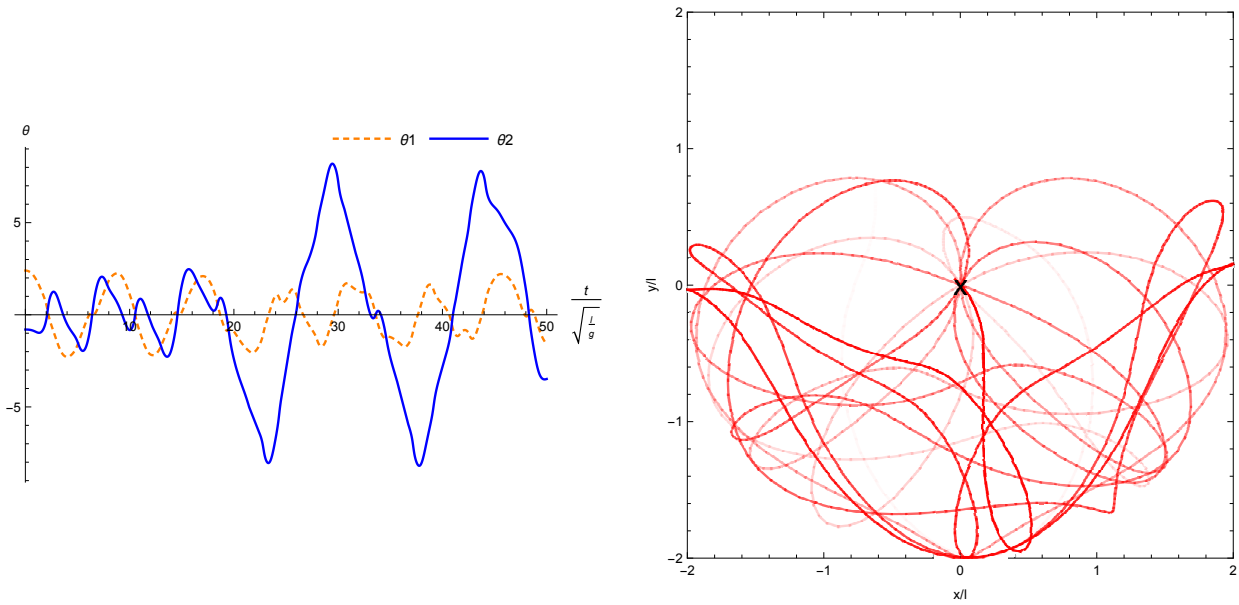
$\theta_1(0) = 0.6,\ \theta_2(0) = -0.2$: still looks like the sum of two normal modes.



$\theta_1(0) = 1.2,\ \theta_2(0) = -0.4$: something much more complicated.



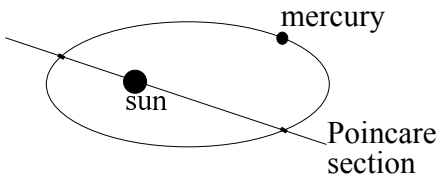
$\theta_1(0) = 1.8,\ \theta_2(0) = -0.6$: lower rod starts to flip.



$\theta_1(0) = 2.4,\ \theta_2(0) = -0.8$

10.5.3 Poincare sections

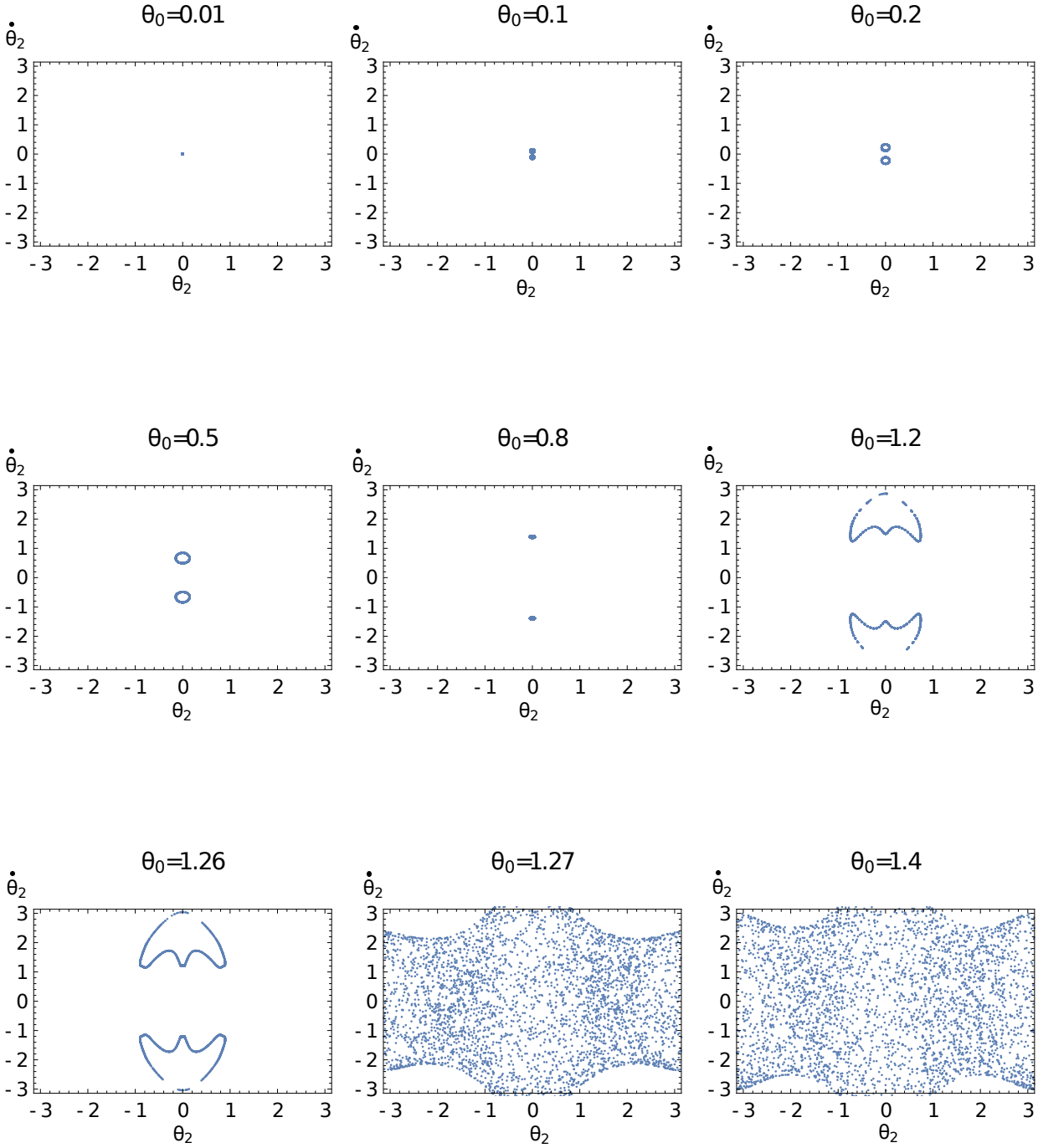
Inspired by the simple pendulum, we would like to make a "phase-portrait" for the double pendulum, to completely characterize all its possible motions. In principle this is a great idea, in practice there is a problem: the phase portrait would live in four dimensions $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ whereas these notes are on two dimensional paper.



The next best thing is a *Poincare section*. It helps to think about a simple problem first. Consider the orbit of Mercury around the Sun. Here a full phase-portrait would also be four dimensional (r, θ, $\dot{r}$, $\dot{\theta}$). To make a Poincare section, we consider the actual orbit of Mercury, and look at a plane (or "section") through the orbit as shown in the diagram. Mercury passes through this section twice every orbit, and we mark a dot on the section every time it passes through. Within the two-body theory of orbits, the orbit is a periodic ellipse, so it will always intersect the section at the same two points, and our Poincare section will be a pair of dots. In general, a section consisting of isolated dots is the hallmark of periodic motion. However, in reality the ellipse of Mercury's orbit itself rotates (or *precesses*) by a fraction of a degree a century (574 arc second per Julian century to be precise) because of gravitational tugs from the other planets. This "precession" will cause subsequent dots on our Poincare section to not exactly line up: instead, over time, the points will trace out a out a line spanning Mercury's largest and smallest approaches to the Sun. A Poincare section that traces a line is the hallmarks of *quasi-periodic* motion.

Applying a similar idea to the double pendulum, we consider a single trajectory, and whenever $\theta_1 = 0$ we plot its 2-D state θ_2 and $\dot{\theta}_2$. This gives us a 2-D plot (θ_2, $\dot{\theta}_2$) which gets a dot every time the motion passes through $\theta_1 = 0$. There are many such Poincare sections, as we get one for each possible motion of the pendulum. Here we show a range for motions with the initial conditions $(\theta_1(0), \theta_2(0), \dot{\theta}_1(0), \dot{\theta}_2(0)) = (\theta_0, \theta_0, 0, 0)$.

For small θ_0 the section contains just two small loops. Lines on a Poincare section are the hallmark of quasiperiodic behavior, as we would expect for a combination of two incommensurate normal modes. The lines indicate the system isn't quite periodic (otherwise it would always come back to the same point, and we would just have dots) but is nevertheless doing something rather simple and predictable. As θ_0 increases, the loops get bigger and more complicated. Then, quite suddenly, when θ_0 increases further, beyond around $\theta_0 = 1.27$ the lines and loops suddenly completely break down and the Poincare section becomes a messy dense set of points with no obvious structure. This indicates a motion that isn't even quasiperiodic: the pendulum keeps swinging in a complex non repeating motion. This motion is called "chaotic" although there is nothing random about it: the pendulum is moving on a completely pre-defined and computable trajectory, it just happens to be very complicated.



Poincare sections for the double pendulum, showing the onset of chaos around $\theta_0 = 1.27$.

10.5.4 Flip time

A different way to get a handle on the pendulum's motion is to plot the "flip time" T_{flip} as a function of $\theta_1(0)$ and $\theta_2(0)$, defined as the time from the start of the motion for the lower rod to flip, or pass through $\theta_2 = \pm\pi$. If we run many RK4 integrations for different values of $\theta_1(0)$ and $\theta_2(0)$, and in each case run until the first flip to compute T_{flip}, we can construct the plot overleaf.

The key message of this plot is how complicated it is: the value of T_{flip} can vary by orders of magnitude with a tiny change in $\theta_1(0)$ and $\theta_2(0)$. This rapidly varying property is particularly obvious in the large central black section with $T_{flip} = \infty$, where the pendulum never flips. This region includes the simple eye-shaped region

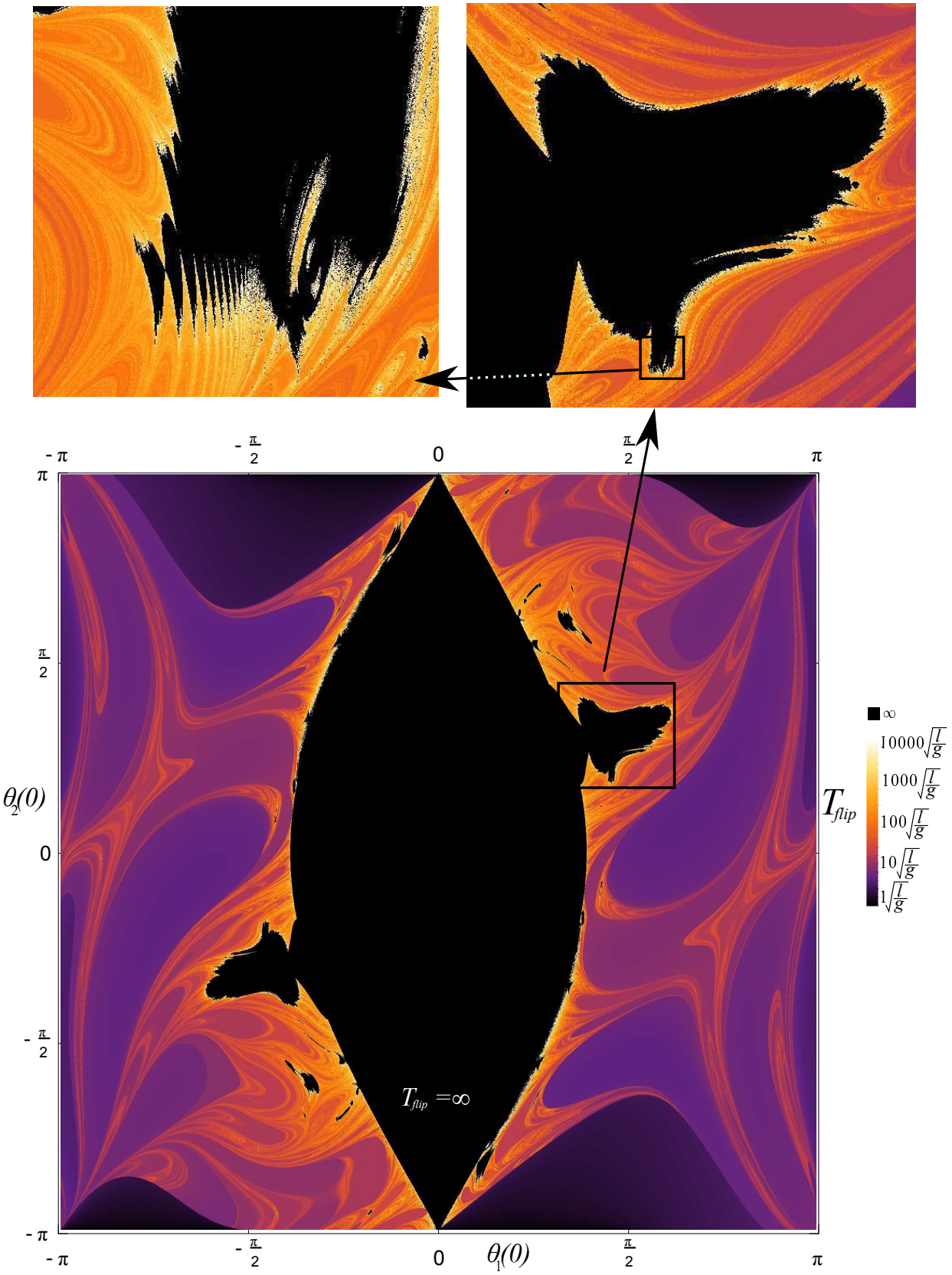
$$3\cos(\theta_1(0)) + \cos(\theta_2(0)) < 2,$$

where (as you showed on examples sheet 3) the total energy of the pendulum is insufficient to flip. However, the full $T_{flip} = \infty$ region is bigger than this, and a much much more complicated shape. Indeed, if you zoom in on the boundary, you discover that the shape is unbelievably complicated, with more and more fine detail and roughness the more you zoom in. Without going into too many details, the shape of the $T_{flip} = \infty$ region is a fractal, which remains complicated and detailed no matter how far you zoom in.

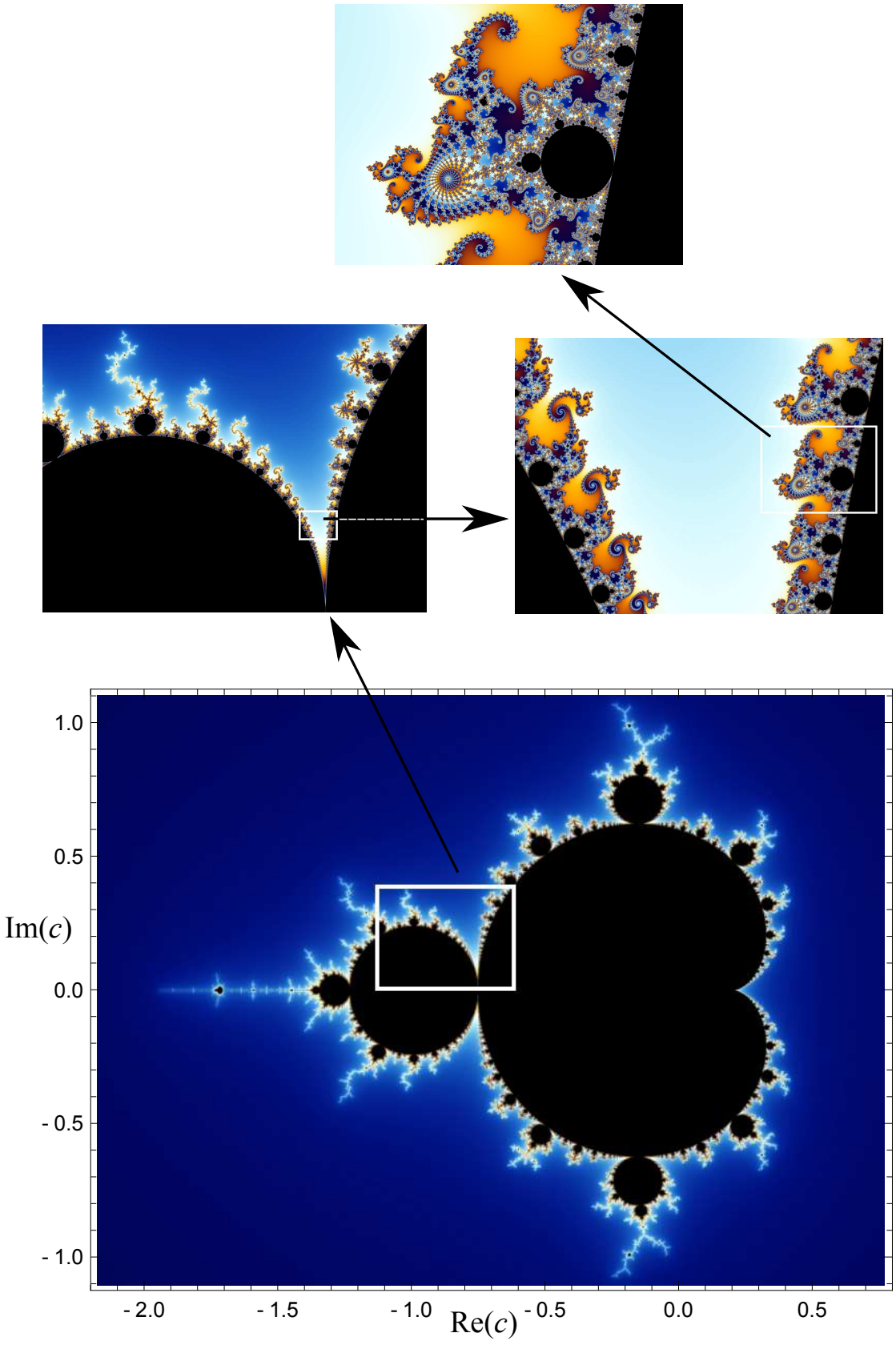
A much more famous fractal is the iconic Mandelbrot set, which is a set of complex numbers occupying a region of the complex plane. To test whether a given number c is in the set, you set $z_0 = 0$, then iterate with the rule

$$z_{n+1} = z_n^2 + c.$$

If the result of this iteration diverges as you iterate, c is not in the Mandelbrot set; if it remains bounded, c is in the Mandelbrot set. This is evidently analogous to the double pendulum, with the iteration being like the computed motion, and divergence corresponding to flipping. Most images of the Mandelbrot set colour the set itself black, and then color points not in the set according to how quickly their iterations diverge. As is clear from the image (doubly overleaf), the black central region (the set itself) has a rough fractal boundary, containing smaller copies of the entire set, which themselves contain even smaller copies on their boundaries.

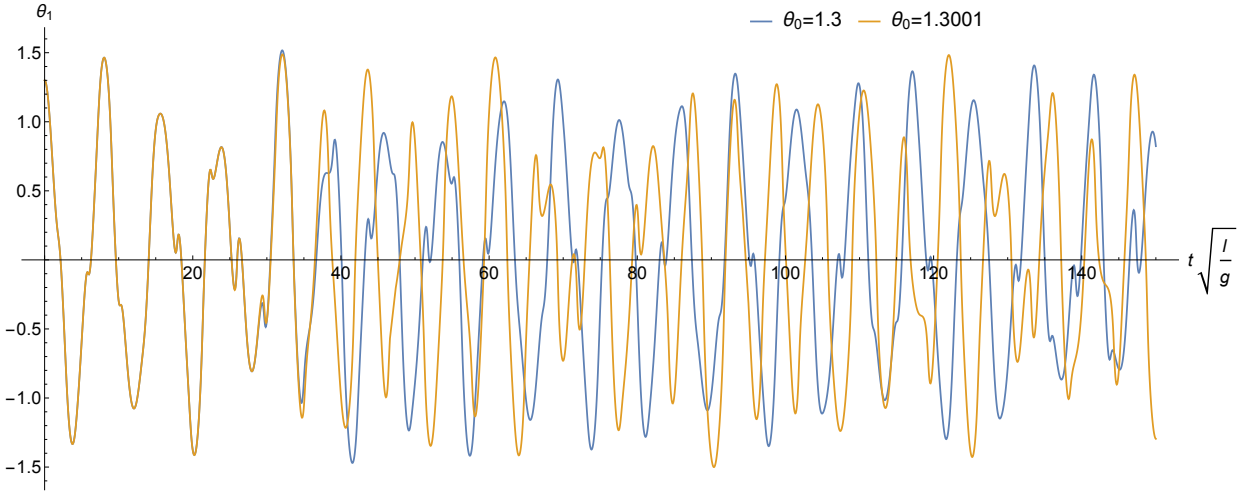


Flip time of the double pendulum when released at rest from $\theta_1(0)$ and $\theta_2(0)$.



The Mandelbrot set (black) is a fractal region of the complex plane. The remainder of the plane is colored by divergence rate. Images from Wolfgang Beyer via wikipedia: CC BY-SA 3.0, https://creativecommons.org/licenses/by-sa/3.0/

10.6 Chaos!



Two trajectories for the double pendulum, which start very similar but quickly diverge.

A consequence of the rough fractal nature of the $T_{flip} = \infty$ region is that there is an extended region around this boundary where, even if you know exactly the value of T_{flip} for a given starting conditions, you don't know much at all about the value of T_{flip} for even very slightly different starting conditions. If you have any uncertainty about the exact starting conditions, you have no hope of making any sensible prediction for T_{flip}. This incredible sensitivity to initial conditions is the defining feature of "chaotic" behavior. Indeed, even away from the boundary, if we plot the $\theta_1(t)$ trajectories for two motions with almost identical starting conditions (above) we see that they quickly diverge and look completely different.

This type of behavior was first noticed in weather prediction by Edward Lorenz, an American meteorologist. He was using a computer to integrate a complicated set of differential equations to forecast the weather. One day he decided to run his weather prediction algorithm a second time, starting at a mid-point from an initial run, and was shocked to discover that, after a couple of simulation months, the predicted weather pattern was completely different. He realized his mistake was that he had only entered the mid-point starting state to the rounded off accuracy of his printer, rather than the full internal precision of his computer, and this initial discrepancy had grown, in his words "the differences more or less steadily doubled in size every four days or so, until all resemblance with the original output disappeared somewhere in the second month." He summarized chaotic behavior as being:

"When the present determines the future, but the approximate present does not approximately determine the future. "

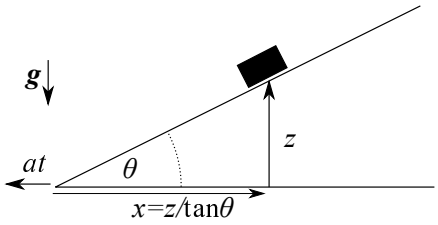
This observation is very troubling. It suggests that if we don't know the initial conditions exactly, long term forecasting in chaotic systems is impossible. To use the traditional poetic language, it seems "the flap of a butterflys wings in Brazil set off a tornado in Texas." In general, it is hard to predict which equations of motion will produce chaotic motion, but chaotic motion is not uncommon in systems with non-linear dynamics equations: other examples include the damped driven pendulum, the three-body gravity problem and financial models of the economy. The general message is, when integrating such equations, it isn't enough to start from a single initial condition and see what happens, you must also start from a few nearby initial conditions and see if they give similar answers.

11 Time variation

A key advantages of Lagrangian mechanics is that it allows us to calculate the equations of motion for a mechanism even when the constraints are changing in time. There are no adaptations required: one still simply forms $\mathcal{L} = T - V$ in terms of suitable coordinates q_i and then calculates the equations of motion in the same way as before: indeed right from the start we included notationally the possibility that $\mathcal{L}$ depends on t (time), but in this section we are actually going to look at some real examples. The only thing to be aware of is that, if $\mathcal{L}$ explicitly depends on time, this indicates the system is externally driven in some way, and we then do not expect the total energy, $E = T + V$, to be conserved during the motion. This expectation matches the Noether theorem result that time invariance leads to energy conservation.

11.1 Simple example: mass on an accelerating wedge

Consider a mass sliding on a wedge, which makes an angle θ with the horizontal. The wedge itself is accelerating in the horizontal direction at velocity $v_w = at$. What value of a is needed for the mass to remain stationary relative to the wedge?



The mass has one degree of freedom, which we describe via its height z. As shown in the diagram, in Cartesian coordinates centered on the wedge tip, the position of the mass is $(x, z) = (z/\tan\theta, z)$, so its velocity relative to the tip $(v_x, v_z) = (\dot{z}/\tan\theta, \dot{z})$. Since the wedge is moving horizontally at $v_w = -at$, the inertial horizontal velocity of the mass is $\dot{z}/\tan\theta - at$, and the Lagrangian is

$$\mathcal{L}(z, \dot{z}, t) = \frac{1}{2}m\left(\dot{z}^2 + \left(\frac{\dot{z}}{\tan\theta} - at\right)^2\right) - mgz,$$

which does indeed explicitly depend on time. We work out the generalized force and momentum in the normal way,

$$p_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z} + m\left(\frac{\dot{z}}{\tan\theta} - at\right)\frac{1}{\tan\theta} \qquad F_z = \frac{\partial \mathcal{L}}{\partial z} = -mg$$

so the equation of motion is

$$m\frac{d}{dt}\left(\dot{z} + \left(\frac{\dot{z}}{\tan\theta} - at\right)\frac{1}{\tan\theta}\right) = -mg \qquad \Longrightarrow \qquad \frac{\ddot{z}}{\sin\theta^2} - \frac{a}{\tan\theta} = -g.$$

The only new subtlety here is that we have to remember the explicit time dependence in p_z when applying the $\frac{d}{dt}$. We see the mass will remain stationary relative to the wedge if $a = g\tan\theta$, i.e. if the total vector acceleration $\mathbf{a} - \mathbf{g}$ has no component down the slope. If the wedge accelerates faster than this, the mass will actually go up the slope.

11.2 Forced Oscillations

11.2.1 Mass on a spring: how to incorporate external forces

In our first example, the time dependence in the Lagrangian arose because the wedge was accelerating, meaning the constraints on the particle were changing in time. A second way the Lagrangian can have time dependence is if the potential energy of the system is a function of time. This arrises naturally if the system is subject to external forces which vary in time.

For example, if we have a 1-D mass on a spring, the Lagrangian is

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2,$$

and the equation of motion is

$$m\ddot{x} = -kx.$$

If the spring is subject to a time varying external driving force $f(t)$, we know this should appear in the equation of motion as

$$m\ddot{x} = -kx + f(t),$$

which will happen naturally if we modify the Lagrangian to

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 + xf(t).$$

We can understand this as follows. If a mass is subject to a constant force in the x direction F, the force does work xF when the mass moves from the origin to x, and is associated with a potential energy $-xF$. These formula are particularly familiar when the constant force is gravity, giving $F = -mg$ and $V = mgz$. Now the potential energy function associated with $f(t)$ is $V = -xf(t)$, which is a function of time because the underlying force is a function of time. Time varying potential energy is the second way time dependence can enter into a Lagrangian.

If we have a sinusoidal applied force, $f(t) = a_0 \cos(\omega t)$, the equation of motion becomes:

$$m\ddot{x} = -kx + a_0 \cos(\omega t) \qquad \Longrightarrow \qquad \ddot{x} + \frac{k}{m}x = \frac{a_0}{m}\cos(\omega t).$$

You studied this equation at length in the vibrations course last year. To solve it, we write the physical equation as the real part of the complex equation

$$\ddot{z} + \frac{k}{m}z = \frac{a_0}{m}\mathrm{e}^{i\omega t},$$

and find the corresponding complimentary function and particular integral. In more detail, to find the complimentary function, we try substituting the form $z_{CF} = A\mathrm{e}^{ipt}$ which gives

$$A\left(-p^2 + \frac{k}{m}\right) = 0,$$

so our trial solution works provided $p = \pm\sqrt{k/m}$. To find the particular integral, we try the same form but with $p = \omega$ to match the right hand side. Substituting, we see this works provided

$$A\left(-\omega^2 + \frac{k}{m}\right) = a_0 \qquad \Longrightarrow \qquad A = \frac{a_0}{\frac{k}{m} - \omega^2}.$$

The complete solution is then the sum of the complimentary function and the particular integral:

$$z = C_1 \mathrm{e}^{i\sqrt{\frac{k}{m}}t} + C_2 \mathrm{e}^{-i\sqrt{\frac{k}{m}}t} + \frac{a_0}{\frac{k}{m} - \omega^2}\mathrm{e}^{i\omega t},$$

which, returning to the real form, requires

$$x = C_3 \cos\left(\sqrt{\frac{k}{m}}t + \phi\right) + \frac{a_0}{\frac{k}{m} - \omega^2}\cos\left(\omega t\right),$$

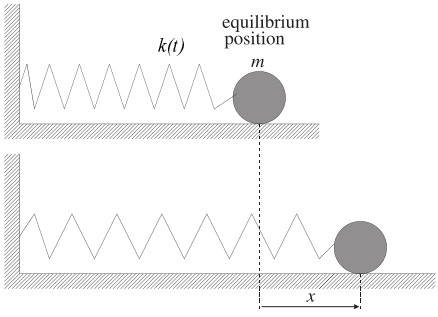
where the constants of integration in the complimentary function, C_3 and ϕ, will be fixed by the initial conditions. Critically, the particular integral diverges if $\omega = \sqrt{k/m}$ indicating the system resonates if the frequency of the driving force matches the natural frequency of oscillation. This resonance is associated with a change in sign of the particular integral, as the motion shifts from being exactly in phase to exactly out of phase with the driving force. In this resonant case, our maths has run into difficulty because the function form of our particular integral is in the complimentary function: we know from IA maths that in this case we should use $z_{CF} = Ate^{i\omega t}$, which will result in a solution that grows linearly in time.

11.3 Parametric Resonance

Many important instances of resonance are not caused by a periodic external driving force, but by a periodic change in the resonating system itself. The classic example is a child swinging: if the child is pushed by a parent we have an external driving force, but a child can also drive their swing themselves by periodically moving their legs. The simplest model to understand this type of behavior is a mass on a spring where the spring constant is a function of time $k(t)$. In practice this could be achieved by having a periodic temperature cycle that changes the underlying elastic properties of the spring material.

The Lagrangian for this system is

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(t)x^2,$$



so the generalized momentum and force are

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad f_x = \frac{\partial \mathcal{L}}{\partial x} = -k(t)x,$$

and the equation of motion, as expected, is simply

$$\ddot{x} = -\frac{k(t)}{m}x,$$

which is a particular example of the general form

$$\ddot{q} = -\omega^2(t)q.$$

To keep things simple we focus on systems where $\omega^2(t) = \omega_0^2(1 + h\cos(\omega t))$, where $h \ll 1$ is small, so the variations in the spring stiffness are rather small, and the equation of motion,

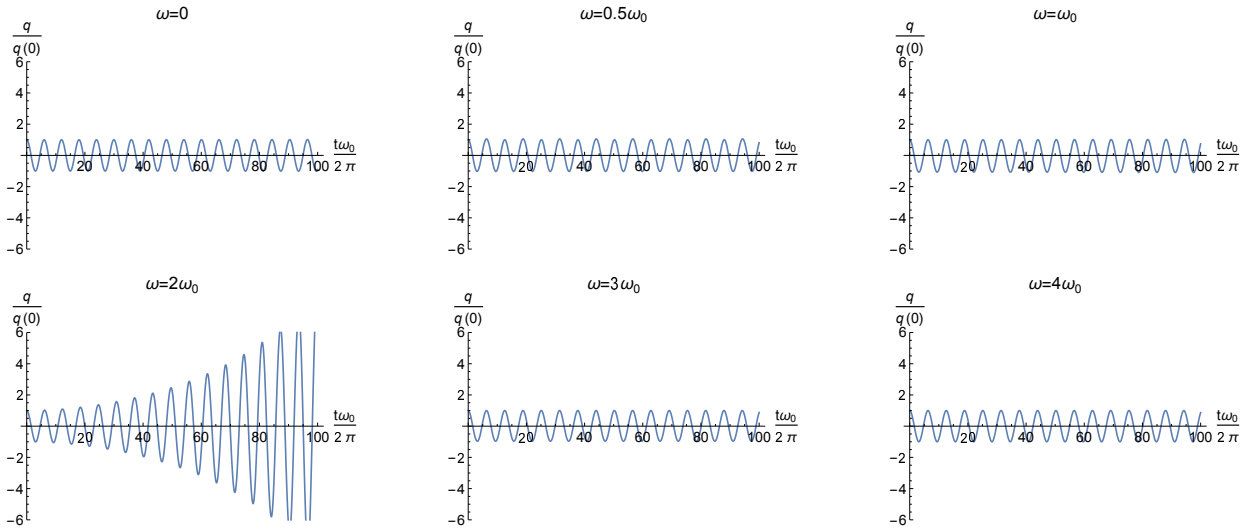
$$\ddot{q} = -\omega_0^2(1 + h\cos(\omega t))q$$

is almost the regular simple harmonic oscillator equation with natural angular frequency ω_0 and solution $q = a\cos(\omega_0 t + \phi)$.

This equation of motion is known as Mathieu's equation, and does not admit a simple closed solutions. However, it is easy to integrate on a computer, to get a sense of how the system behaves.

The first important observation is that if we start from $q = 0$, $\dot{q} = 0$, then the solution is trivial $q(t) = \dot{q}(t) = 0$, and nothing happens. This is very different to the externally driven oscillator from the previous section.

To see anything at all, we thus try numerical integration starting at $q(0) = 1$, $\dot{q}(0) = 0$ and, for definiteness, take $h = 0.1$. As shown below, we quickly discover that, at most values of ω the solution strongly resembles $q(0)\cos(\omega_0 t)$, which would be the motion of the system without the $h\cos(\omega t)$ time varying term being present at all: most of the time the new term is unimportant. This is even true at $\omega = \omega_0$ where we might have expected something interesting like resonance. However, at $\omega = 2\omega_0$, we see something quite different: the motion becomes an oscillation with exponentially increasing amplitude.



Motion or a parametric oscillator for different driving frequencies. Resonance occurs when the driving frequency is twice the natural frequency.

Resonance when the driving is at twice the natural frequency is the hallmark of parametric resonance. In hindsight, we can understand this result by writing the equation of motion for the system in the form

$$\ddot{q} + \omega_0^2 q = h\omega_0^2\cos(\omega t)q, \tag{2}$$

which is the standard form for a driven oscillator, but with the driving force $f(t) = h\omega_0^2 \cos(\omega t)q$ which is proportional to q. If the system is oscillating at its natural frequency $q = A\cos(\omega_0 t)$, then this driving force term looks like

$$h\omega_0^2 \cos(\omega t)A\cos(\omega_0 t) = \frac{hA}{2}\left(\cos\left((\omega_0 - \omega)\,t\right) + \cos\left((\omega_0 + \omega)\right)t\right).$$

If we have $\omega = 2\omega_0$ then one of these two terms looks like a driving force at the resonant frequency, $\omega - \omega_0 = \omega_0$, which causes the amplitude to grow. Since the size of this force is proportional to amplitude, we get an exponential increase in amplitude. The $\omega + \omega_0 = 3\omega_0$ term is off resonance, and has little impact on the motion.

In contrast, if we drive at the natural frequency, $\omega = \omega_0$, neither term would be on resonance, and the system does not resonate. Resonance at $2\omega_0$ and exponential growth of amplitude are the hallmarks of "parametric" resonance. In contrast, in traditional resonance with a driving force, one finds resonance at ω_0, and the amplitude grows linearly in time.

If one thinks through a cycle of oscillation for a mass on a spring, one can see why having stiffness variation at $2\omega_0$ leads to a growing amplitude response. In this scenario, the mass feels a lower spring force when it is moving away from the equilibrium point, and a larger spring force when it is being pulled back, which pumps energy into the system and causes the amplitude to increase. The work done by the varying force on the mass in one cycle, $\int h\cos(2\omega_0 t)\omega_0 A\sin(\omega_0 t)\mathrm{d}t$ is positive, showing the variation pumps energy into the system. In contrast, if the spring varies at ω_0, it pushes harder for a half cycle, then softer for a half cycle, so the net work done $\int h\cos(\omega_0 t)\omega_0 A\sin(\omega_0 t)\mathrm{d}t$, vanishes and the amplitude does not grow.

11.3.1 Pendulum with changing length (non examinable)

Consider a pendulum whose length $L(t)$ is a function of time, as might occur if the string winds over a pulley at the pivot and then connects to a motor. In this case, the Lagrangian will be

$$\mathcal{L}(\theta, \dot\theta, t) = \frac{1}{2}m(L(t)^2\dot\theta^2 + \dot L(t)^2) + mgL(t)\cos\theta,$$

the generalized momenta and force are

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot\theta} = mL(t)^2\dot\theta, \qquad F_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = -mgL(t)\sin\theta,$$

and the equation of motion is

$$\frac{d}{dt}\left(L(t)^2\dot\theta\right) = -gL(t)\sin\theta \quad \rightarrow \quad L(t)^2\ddot\theta + 2L(t)\dot L(t)\dot\theta = -gL(t)\sin\theta.$$

This is more complicated than the basic parametric resonance equation, but the motion in fact behaves in an almost identical way. If we write $L(t) = l(1 + h\cos\omega t)$ and expand assuming h, θ, $\dot\theta$ and $\ddot\theta$ are all small then, to first order we get the equation of a simple pendulum, which produces simple harmonic motion for θ:

$$l^2\ddot\theta = -gl\theta, \qquad \rightarrow \qquad \theta = \theta_0\cos\left(\sqrt{\frac{g}{l}}t + \phi\right).$$

If we continue expanding, to second order we get

$$l^2\ddot{\theta} + 2l^2 h\cos\omega t\ddot{\theta} - 2\dot{\theta}l^2 h\omega\sin\omega t = -gl(1 + h\cos\omega t)\theta.$$

This equation has three new "driving" terms, that are individually very similar to the parametric driving term in eqn. (2). In particular, if the pendulum is undergoing a motion close to $\theta = \theta_0\cos\left(\sqrt{\frac{g}{l}}t\right)$ then each driving term is proportional to $h\theta_0$, and contains a product of a trig term at ω and a trig term at $\sqrt{\frac{g}{l}}$. Each trig product breaks down into components at the sum and differences of the two frequencies, (via trig identities such as $\cos A\cos B = \frac{1}{2}\left(\cos\left(A-B\right) + \cos\left(A+B\right)\right)$) so each term produces a difference component that will drive a resonance when $\omega = 2\sqrt{\frac{g}{l}}$. Thus the variable length pendulum also resonates at twice its natural frequency and, since these driving terms are proportional to θ_0, the resonance amplitude will grow exponentially: the motion has both the hallmarks of simple parametric resonance.

This leads to an interesting effect in a carefully tuned "spring-pendulum" (pendulum consisting of a mass on a spring). As you showed on the examples sheet, this system has two normal modes, a bouncing spring mode and a swinging pendulum mode. Imagine tuning the system so that the bouncing mode has twice the frequency of the swinging mode. If you then start the mass bouncing, the bounce causes the pendulum to change length at twice its natural frequency, driving a parametric resonance of the swinging mode. As I will demonstrate in lectures, as time progresses, the energy of the motion will swop from the bouncing mode to the pendulum mode, until the system is purely swinging without bouncing. If you wait long enough, the energy will then swop back to the bouncing mode. We see that the two modes of vibration are coupled, and our typical analysis of independent normal modes has failed. Generically, this type of effect is driven by higher-order (non linear) terms in the equations-of-motion that we discard in a normal mode analysis, and these higher order terms lead to many interesting effects: mode-coupling, multi-stability, self oscillation, chaos, amplitude-death, solitons, limit-cycles and much more besides. Much of this goes under the general heading of "nonlinear vibrations", which is also the title of 4C7.