Senior Physics Challenge 2010 – Introduction to Special Relativity

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1 Absolute position and the geocentric universe

The geocentric model of the universe (Figure 1) places the Earth at the center of the universe while the Sun, Moon, planets and stars go around it. This was the dominant model of the universe until the renaissance when more accurate measurements of the motions of the planets made it clear that the planets, including the Earth, rotate about the Sun — Figure 2. In a geocentric universe it makes sense to talk about the absolute position of an object. We can set up a special coordinate system based on the centre of the Earth and ascribe physically meaningful coordinates to every point in space. This means that in our geocentric universe we can reasonably expect the laws of physics to depend on position — maybe for example there is a force that pulls everything towards the centre of the Earth (that is, to the centre of the Universe), and the strength of the force depends on how far out from the centre you are (your absolute position). In this case, if you find yourself in a sealed box at some point in space so you can't see out, you can still work out how far you are from the centre of the universe by measuring the inwards force on a particle. By looking at how the force varies in time you could also deduce whether the box you are in is moving. This is the sense in which, in a geocentric universe, both position and velocity are absolute — even in a sealed box with no windows you can deduce both your position and your velocity by conducting some experiments.

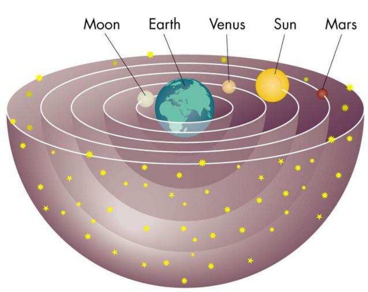


Figure 1: A geocentric universe

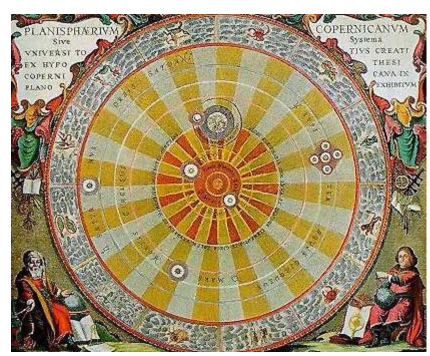


Figure 2: A heliocentric universe

At first glance the heliocentric model of the universe, which puts the Sun at the centre and has the Earth and planets revolving around it, is just as absolute as the geocentric model. The only difference it that the special coordinate system is based on the Sun rather than the Earth. However, the Earth is moving around the Sun at great speed — about 30km/s — and yet on Earth it is very difficult to detect this. For example, we might think we could tell by throwing a ball straight up into the air and watching where it lands. If the ball flies for 1s, the Earth should have moved 30km during the balls flight, so we might have expected the ball to land several kilometers from where we threw it, but this does not happen. Nor do we feel ourselves being pulled towards the centre of the universe. During the renaissance these objections lead to the heliocentric model being very controversial — if the Earth isn't at the centre of the universe and is moving fast, why can't we tell?

2 The principle of relativity

To counter these objections to the heliocentric universe, Galileo pointed out that even on Earth, if you are below deck on a ship, it is impossible to detect whether the ship is moving without looking out of the porthole. Specifically he wrote:

> "Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it. With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction. When you have observed all these things carefully (though doubtless when the ship is standing still everything must happen in this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than toward the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite. The droplets will fall as before into the vessel beneath without dropping toward the stern, although while the drops are in the air the ship runs many spans. The fish in their water will swim toward the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air. And if smoke is made by burning some incense, it will be seen going up in the form of a little cloud, remaining still and moving no more toward one side than the other. The cause of all these correspondences of effects is the fact that the ship's motion is common to all the things contained in it, and to the air also. That is why I said you should be below decks; for if this took place above in the open air, which would not follow the course of the ship, more or less noticeable differences would be seen in some of the effects noted."

The essential observation here is that you can't conduct an experiment to determine where you are or how fast you are moving. Since our experiments are governed by the laws of physics, this means that the laws of physics must also not depend on where you are or how fast you are moving. This is an experimental observation, not a philosophical deduction. We encapsulate this thought in the **principle of relativity** which we can write informally as

> "if you are stuck in an isolated box with no windows you can't tell how fast the box is moving or where the box is"

or more formally as

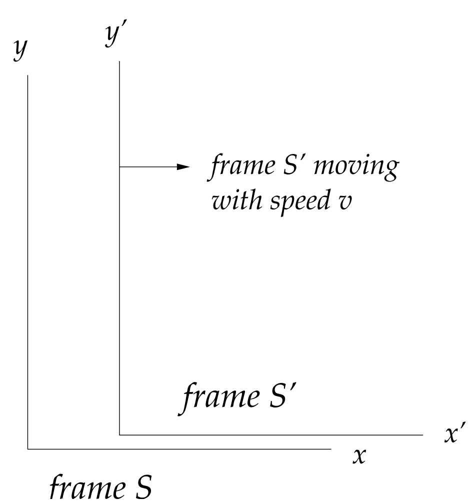
> "The laws of physics are the same in all inertial frames'.'

We define an inertial frame to be one in which free particles move in straight lines. It is important to notice that the principle of relativity applies to position and velocity, but not to to linear acceleration or rotation. This is because, as anyone who has been in a plane at take-off or sat on a roundabout will know, it is perfectly possible to detect that you are rotating or accelerating.

3 Galilean relativity and Newton's Laws

To probe whether a physical theory obeys the principle of relativity described above, we have to establish how the same events are seen by observers who are at different places and moving at different speeds, so that we can see whether the different observers can explain the events using the same physical laws. To describe a set of events, an observer must say where and when the various events happened. To do this each observer sets up a coordinate system consisting of an x, y and z axis so that they can record where the evens happen, and a clock so that they can record when the events happen. We give all our observers a standard meter ruler to mark the scale on their spatial axes and a standard clock to measure time with.

We typically consider two observers who set up coordinate systems S and S', such that S' is moving at a speed v with respect to S. The coordinate systems are in the "standard" configuration, meaning the x axis in S is parallel with the x axis in S' (which we call the x' axis), and similarly for the y and z axes. Furthermore, the relative velocity between the two coordinate systems is in the x direction, and both clocks are started at the moment the two origins are in the same place, so if the observer in S sees an event at $x = y = z = t = 0$, then the observer in S' also sees it at $x' = y' = z' = t' = 0$. This is illustrated in the figure below.



If the observer in the frame S observes an event with spatial coordinates (x, y, z) at time t, where and when is it according to the observer in S'? The Galilean answer to this question is simple and intuitive. Both observers started their clocks simultaneously, so the time on the clock in S', which we call t' will be the same as the time on the clock in S. The frame S' will have moved a distance vt in the x direction since the origins were in the same place, as pictured in Figure 3, so the coordinates (x', y', z') observed in the S' frame will be

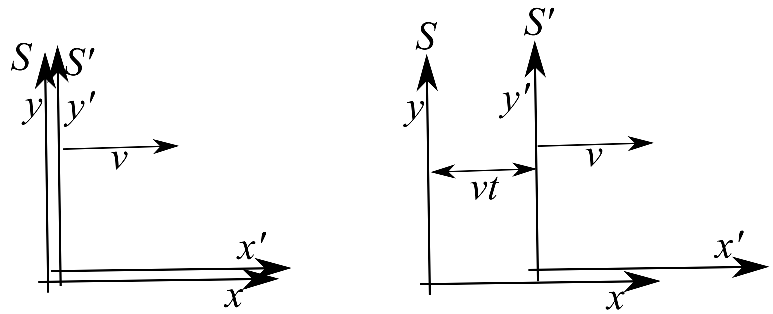


Figure 3: Left: Two frames S and S' in the standard configuration at $t = t' = 0$, at which point the two frames lie exactly on each other, although S' is moving at a speed v with respect to S. Right: A time t later S' has moved a distance vt in the x direction.

$$x' = x - vt \tag{1}$$
$$y' = y \tag{2}$$
$$z' = z \tag{3}$$
$$t' = t. \tag{4}$$

Conversely, if we know the primed coordinates observed in S', the coordinates according to the observer in S will by

$$x = x' + vt \tag{5}$$
$$y = y' \tag{6}$$
$$z = z' \tag{7}$$
$$t = t'. \tag{8}$$

The Newtonian laws of physics obey the principle of relativity using these Galilean transformations. This is simple to see — Newton's third law states that the acceleration of a body ($\mathbf{a}$) is given by the force the body is subject to ($\mathbf{F}$) divided by the mass of the body (m). Taking the derivatives of the transformation laws with respect to time (since $t = t'$ there is nothing to worry about here) we see that, since v is constant,

$$\dot{x}' = \dot{x} - v \tag{9}$$
$$\dot{y}' = \dot{y} \tag{10}$$
$$\dot{z}' = \dot{z} \tag{11}$$
$$\tag{12}$$

which are the Galilean velocity transformation laws. Taking a second derivative to get acceleration transformation laws gives,

$$\ddot{x}' = \ddot{x} \tag{13}$$
$$\ddot{y}' = \ddot{y} \tag{14}$$
$$\ddot{z}' = \ddot{z}. \tag{15}$$
$$\tag{16}$$

This means the acceleration of a particle observed in S will be the same as the acceleration of the particle observed in S'. Furthermore, the distance between two particles observed in S is the same as the distance between the two particles observed in S'. Therefore, if an observer in S can predict the forces on a group of particles by using force laws that only depend on the distances between the particles (such as Newton's law of gravity), and apply $\mathbf{F} = m\mathbf{a}$ to work out the accelerations, and observes that the predicted accelerations match the observed accelerations, then so can an observer in S' watching the same events. This is clear since the observer in S' sees the particles separated by the same distances, so he calculates the same forces, and he also observes the same accelerations. If Newton's laws of motion are obeyed in one frame, it will also be obeyed in the other. This means that Newtonian physics already obeys the principle of relativity[1].

4 Electromagnetism and the speed of Light

Significant problems with classical physics first arose with the discovery of the laws of electromagnetism by Maxwell. Maxwell was able to write down four equations that described all the behavior of electric and magnetic fields, and his equations made a striking prediction — there should be traveling electro-magnetic waves. The speed of these waves, c, is predicted by the equations, and matches the observed speed of light, so Maxwell concluded that light was an electro-magnetic wave.

The difficulty reconciling electromagnetism with the principle of relativity arises because it is very unclear what the speed predicted by the equations is with respect to. If we derive a theory of sound waves in air, it is clear that the speed we predict is with respect to the stationary bulk of the air. If we derive a theory of bullets coming out of a gun, it is clear that the speed we derive for the bullets is with respect to the gun. The simplest interpretation of the speed of electro-magnetic waves is that it is with respect to some supporting medium called the "ether". This would mean that it is possible for an observer to detect whether he is moving with respect to the ether by measuring the speed of light. Consider the following scenario (illustrated in Fig. 4), a stationary alien (A) fires a laser pulse at Bob (B) who, rather naturally, is running away at a speed v. What is the speed of the laser pulse with respect to the Bob and the Alien? Assuming the Alien is stationary with respect to the ether, the light will propagate through the ether and away from the Alien at the full speed of light c. The speed of the pulse with respect to the Alien is therefore c. However, Bob is running away from the pulse at a speed v, so the speed of the pulse relative to Bob should only be $c - v$. If Bob is fast enough the laser pulse will take a very long time to catch him up, if he is faster than the speed of light it never will. If Bob treats this situation as an opportunity to measure the speed of light, he will record the answer $c - v$ and hence know that he is moving with respect to the ether because this is slower than c.

Question: What would the speed be with respect to Bob and the Alien, if the laser was replaced by a standard gun that ejected bullets with a speed c? What if it was a sound gun and Bob is stationary in the Air?

[1]Notice that we have assumed that all observers agree on the mass of an object.

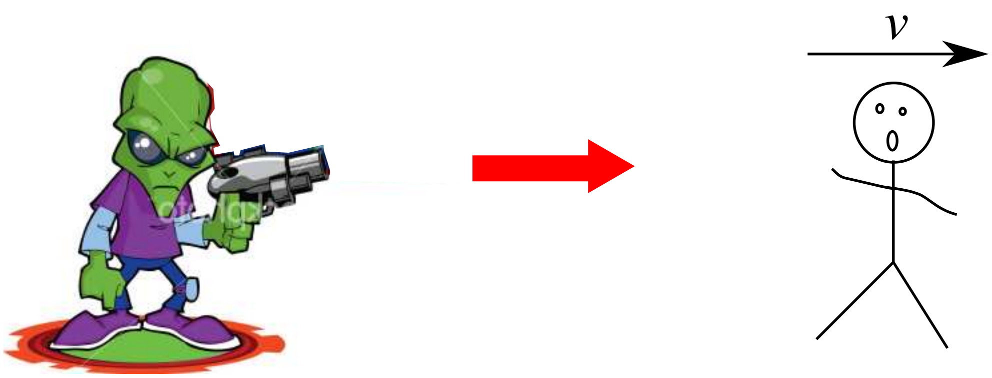


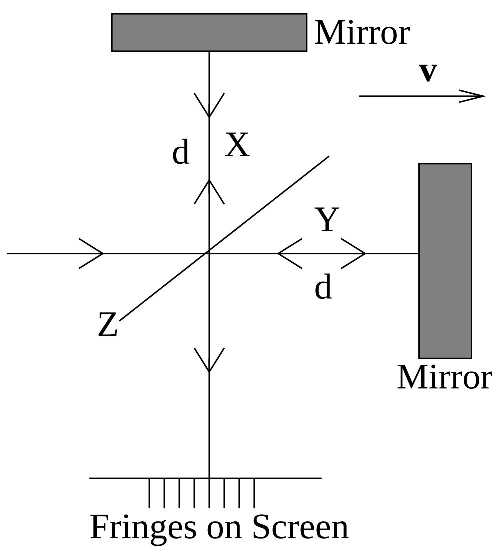
Figure 4: A stationary Alien fires a laser pules at Bob, who is running away at a speed v.

Question: What are the transformation laws for velocities in Galilean Relativity?

The problem with the ether conjecture is that, experimentally, the speed of light does not depend on the speed of the observer. In the above scenario, the light approaches Bob at a speed c with respect to Bob and departs from the Alien at a speed c with respect to the Alien, making running away a pretty fruitless exercise. This defies our Newtonian intuition since it means that light does not obey the Galilean rule for transforming velocities, but it is pleasingly relativistic — it is not possible to use the observed speed of light to work out whether you are moving or not.

Michelson and Morley experiment

It is fair to ask how we know that the speed of light does not depend on the speed of the observer. The historically decisive experiment was conducted by Michelson and Morley. In their experiment, pictured below, a beam of light is split into two perpendicular beams using a half silvered mirror (Z). These perpendicular beams (X and Y) propagate in their given directions until they are reversed by a pair of mirrors and returned to the half silvered mirror (Z). Half the light from each of these beams is then directed by the half silvered mirror towards a screen where interference fringes between the two beams are observed. The key idea is that the Earth is moving round the



Sun at a very substantial speed, and therefore presumably moving through the ether. The length of time it takes the light to travel up and down one of the arms of the apparatus will depend on the angle between the arm in question and the velocity the apparatus is making through the ether (calculating the time difference is one of the questions on the examples sheet), so the X and Y beam will take different amounts of time to reach the screen. If the apparatus is rotated this time difference will change and therefore the interference pattern observed on the screen will change. This is not observed, leading us to conclude that either the apparatus is not moving through the ether or there is no ether. The first possibility is largely eliminated by doing the experiment again 12h later, so the direction of the Earths rotational velocity has reversed, and 6 months later so that the direction of the Earths orbital velocity has reversed.

5 Einstein's Postulates

Einstein proposed that the principle of relativity was correct and that the Maxwell laws of electromagnetism were fundamental physical laws, which lead him to make two postulates:

1. the laws of Physics are the same in all inertial frames;
2. the speed of light is the same in all inertial frames.

The theory of special relativity that he introduced allows both these postulates to be held simultaneously by modifying the Galilean transformation laws discussed above.

6 Time dialation

One immediate consequence of Einstein's postulates is that observers moving at different speeds with respect to each other do not agree on the time interval between two events. We can see this by considering a clock consisting of a pulse of light bouncing between two mirrors — see Figure 5.. One tick of the clock is the time for the light to go right round the clock, which is to say the time for it to travel from one mirror to the other, be bounced of the second mirror and return to the first. If the length between the mirrors is ℓ then this will take a time $t = 2\ell/c$.

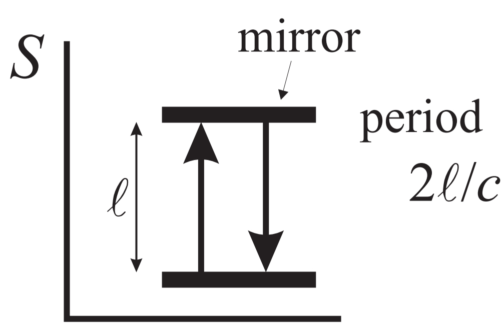


Figure 5: Measuring time using a light clock — a beam of light bouncing between two mirrors.

We now consider two identical light clocks, one at rest in a frame S and one at rest in a frame S' which is moving at a speed v with respect to S in the standard configuration. In both frames the time period observed for the clock that is stationary in that frame is still $2\ell/c$. However, if we consider the clock in S' as seen by an observer in S — Figure 7 — the clock is now seen to be moving. This means that the mirrors are not always in the same place when they are struck by the light beam, rather, if the time period of the clock observed in S is t then the top mirror will

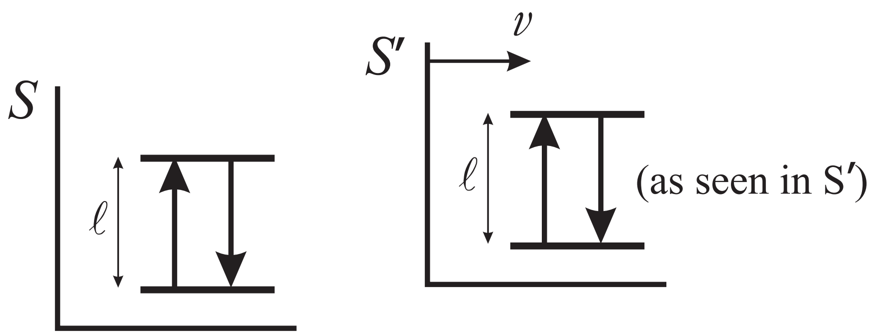


Figure 6: Two light clocks, one (left) stationary in S and one (right) stationary in S', both drawn as seen in the frame they are stationary in.

move a distance $vt/2$ while the light travels from the bottom mirror. This is illustrated below Figure 7.

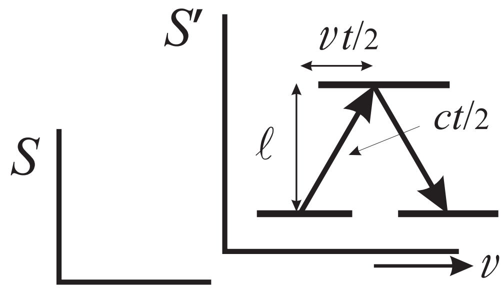


Figure 7: The light clock that is stationary in S' as seen according to an observer in S. The clock moves with a speed v and therefore travels a distance vt in one period of the clock.

The total distance, d, traveled by the light in one period of the clock as observed in S is, by Pythagoras, $d = 2\sqrt{\ell^2 + (vt/2)^2}$. Since according to Einstien's postulate, the speed of light must still be c, the time period observed in S is

$$t = \frac{d}{c} = \frac{2\sqrt{\ell^2 + (vt/2)^2}}{c}. \tag{17}$$

If we square both sides of this equation move all the terms involving t the the left hand side we get

$$\frac{t^2}{4}\left(1 - \frac{v^2}{c^2}\right) = \frac{\ell^2}{c^2} \tag{18}$$

. However, we know that the time period for the clock observed in the S' was $t' = 2\ell/c$, so we can replace the right hand of this equation by $(t'^2)/4$ giving

$$\frac{t^2}{4}\left(1 - \frac{v^2}{c^2}\right) = \frac{t'^2}{4}, \tag{19}$$

We write this result as

$$t = \gamma t' \tag{20}$$

where we have introduced the so called "gamma-factor"

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{(1 - v/c)(1 + v/c)}}. \tag{21}$$

Exercise: Sketch a graph of the gamma factor as a function of velocity v.

This is a remarkable result. The two observers measure a different time interval between the same ticks of the same clock if they are moving at different speeds. This result is known as time dilation since the time between ticks appears to be longer in S (γ is always greater than one), leading us to the general statement that moving clocks run slow according to stationary observers. We call the time between ticks as measured in the clocks stationary frame the **proper time** between the two events (ticks), which we normally write as T_0. The proper time between two events is the time between them as measured in a frame in which they happen in the same place, and is the shortest time that can be measured between them. The time between them measured in a different frame moving at a speed v is

$$T = \gamma T_0. \tag{22}$$

Although our analysis made use of a very specific light clock, this result must hold for any pair of events since, if the time between any pair of events did not behave in this way, we could use use a light clock to measure the discrepancy and hence infer that we were moving.

6.1 Mysterious muons

The γ factor is very close to one unless the velocities involved are similar to the speed of light, so time dilation is not noticed in day-to day life. One close-to-home phenomenon that depends on time dialation is the arrival of muons (a type of elementary particle similar to but heavier than an electron) generated in the upper atmosphere at the surface of the Earth. The muons are generated when cosmic rays strike the upper atmosphere about 60km above the surface of the Earth, as shown below.

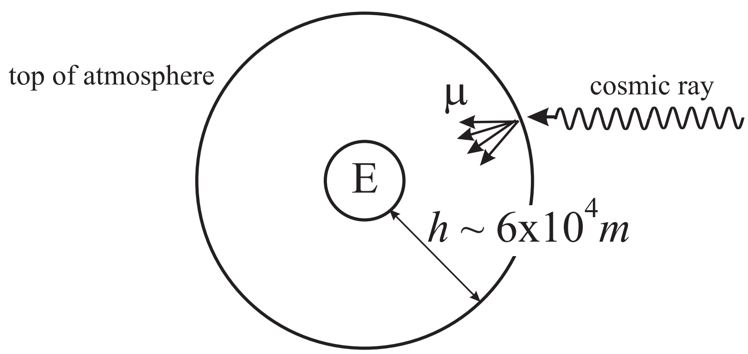


Figure 8: Muons are generated when cosmic rays strike the upper atmosphere. The they travel through the atmosphere to the surface of the Earth. Not to scale!

They are created with a huge amount of energy, so they travel towards the Earth at a speed very close to the speed of light and their travel time is approximately $h/c = 200\mu$s. However, when we make muons in the lab they decay with a half life of $\tau_0 = 1.5\mu$s, so the journey to the surface of the Earth takes 133 half lives, and therefore and we should only expect 1 in 2^{133} muons to get through. In fact about 1 in 8 gets through, so there is an easily measurable flux of muons at the surface of the Earth.

We can use time dilation to explain how the muons get through the atmosphere. If 1 in 8 get through then, in the muons rest frame, 3 half lives must have elapsed between the muon being created and arriving at the surface ($8 = 2^3$). The time for these three half lives (which are like ticks of a clock) to elapse in the Earth's frame will then be $3\gamma\tau_0$, which must be equal to the time we observe the muons to take traveling through the atmosphere in the Earth's frame, namely h/c. This allows us to find γ and hence the speed of the muons.

Notice that the muons clock appears to be running slowly according to observers on Earth because the muon is moving. As we said before, moving clocks run slow according to stationary observers.

6.2 Doppler shifts

In classical physics, the frequency of a wave depends on the speed of the emitter with respect to the receiver. This is why the pitch of a fire engine's siren appears to change when the engine drives past you. The classical doppler shift is easy to analyze — consider a moving source (a fire engine?) that emits a regular series of clicks separated in time by $T_0 = 1/f_0$ (so f_0 is the frequency of the clicks). The clicks travel through the air with speed c, which here we are using as the speed of sound waves in air. If the source is stationary then the clicks will be spaced in the air by a distance cT_0 because each click will move a distance cT_0 away from the source before the next click is released — see Figure 9. If the source is moving at a speed v away from the observer then it will move a distance vT_0 between the emission of two clicks, so the total distance between the clicks in the air will be $(c + v)T_0$. This is also illustrated in Figure 9.

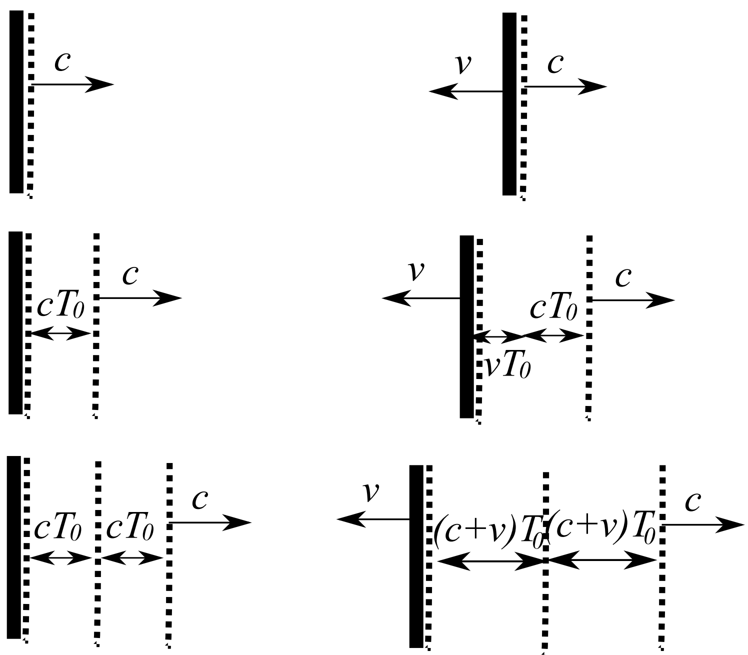


Figure 9: Left: A stationary source (thick black bar) emits a click (dotted line) that moves through the air at a speed c. The three diagrams show the positions of the clicks and the source immediately after the emission of the first second and third click. Right: Exactly the same diagram, but with a source moving at a speed v, increasing the spacing between the clicks.

A stationary observer will here the clicks arrive more slowly if the source is moving away from him. The time between clicks observed by a stationary observer is simply the distance between them divided by the wave speed (c), so the observed frequency in the moving case is

$$f = \frac{c}{T_0(v + c)} = f_0 \frac{1}{1 + v/c}. \tag{23}$$

If the source had been moving towards the observer then the new frequency observed at the origin would be

$$f = \frac{c}{T_0(c-v)} = f_0 \frac{1}{1-v/c}. \tag{24}$$

The above classical analysis is perfectly adequate for fire engines, but breaks down if the source is moving at a speed close to the speed of light. This is because, as well as the classical doppler shift, we also need to take into account the time dilation that the source undergoes because it is moving at a relativistic speed — if the time period between clicks is T_0 in the sources rest frame, then it will be γT_0 in the observers rest frame. If we now assume that rather than a fire engine emitting sound pulses, we have a rocket emitting light pulses, we can use the previous analysis but with an effective time period γT_0 to see that the observed frequency will be

$$f = \frac{c}{\gamma T_0(v+c)} = f_0 \frac{1}{\gamma(1+v/c)} = f_0 \frac{\sqrt{1-v^2/c^2}}{1+v/c} \tag{25}$$

$$= f_0 \frac{\sqrt{(1-v/c)(1+v/c)}}{1+v/c} \tag{26}$$

$$= f_0 \sqrt{\frac{1-v/c}{1+v/c}}. \tag{27}$$

Similarly, if the source is moving towards the origin we have

$$f = \frac{c}{\gamma T_0(c-v)} = f_0 \frac{1}{\gamma(1-v/c)} = f_0 \frac{\sqrt{1-v^2/c^2}}{1-v/c} \tag{28}$$

$$= f_0 \frac{\sqrt{(1-v/c)(1+v/c)}}{1-v/c} \tag{29}$$

$$= f_0 \sqrt{\frac{1+v/c}{1-v/c}}. \tag{30}$$

The frequency of a light source is lowered if it is moving away from us. Astronomers call this "red-shift" and can use it to work out how fast astronomical objects are moving. This is possible because certain atoms are known to emit at certain frequencies/wavelegths which can be measured on Earth, so if a collection of such atoms is seen emitting at a lower frequency it must be because they are moving and have been doppler shifted. The red shift is defined as

$$r = \frac{\lambda}{\lambda_0} = \frac{f_0}{f} = \sqrt{\frac{1+v/c}{1-v/c}}. \tag{31}$$

Question: Why "red-shift" not "blue-shift"? Why are almost all sources moving away from us?

7 Length Contraction

Returning to the atmospheric muons, we have understood that in the Earth's frame the muons travel through the atmosphere at a speed $v \approx c$, causing the muons clock to appear to run slow according to observers on Earth. Applying the time dilation result, if the journey takes a time t_m in the muons rest frame and a time t_E in the Earths rest frame, $\gamma t_m = t_E$. This allows the muon to travel through the atmosphere in a small number of half lives according to its own clock, even though it took many half lives according to clocks on Earth. Suppose we now watch the same events from a frame moving with the muon. The muon is stationary in this frame, and the Earth is heading towards it at a speed v. Since the muon is stationary, in this frame its clock is running at full speed, so how does the muon explain how it got through the atmosphere?

The answer to this problem is simple but profound. During the journey in the muon's frame the Earth moves a distance vt_m, so the atmosphere has height $h' = vt_m$. In the Earths frame, the height of the atmosphere is $h = vt_E = v\gamma t_m$, so the height of the atmosphere in the muons frame is less than the height of the atmosphere in the Earths frame. The two are related by

$$h' = \frac{h}{\gamma}. \tag{32}$$

This means that the atmosphere appears shorter by a facto of γ in the muon's frame because it is moving. The length of the moving atmosphere is shortened by the same amount as the clock on the moving muon ran slow. This is the famous length contraction effect — a body with length L_0 in its rest frame appears to have a length L_0/γ according to observers moving with a velocity v. The contraction only occurs along the direction of the velocity. The longest that an object ever appears is its length in its rest frame, which is called its **proper length**.

8 Lorentz Transformations

We have shown that Einstein's postulates require us to abandon the notion that all observers can agree on the time interval between events or the length of objects — rather observers moving at different speeds will record different answers. However, our reasoning so far has been rather *ad-hoc*. We would really like to write down new transformation laws so that if we know the space-time coordinates of an event in one frame (x, y, z, t), we can find the space-time coordinates observed in another frame (x', y', z', t'). These are the relativistic equivalent of the Galilean transformations we discussed at the start of the course. As before we will restrict ourselves to frames that are in the standard configuration — Figure 3 — meaning that S' is moving at a speed v in the x direction with respect to S and that the clocks in both frames are started at the instant that the origins of the two frames are in the same place.

8.1 Transformation of x

Imagine that a man is standing at rest in S a distance x along the x axis and holding a piece of chalk. At a time t he makes a mark on the axis of the frame S'. The coordinates of this event in S are simply x and t since $y = z = 0$. Where on the x' axis is the mark made? — this will be the x' coordinate of the event. If an event happens at a time t and at a position x in the frame S then the origin of the frame S' is at $x = vt$, so the distance between the event and the origin of S' as measured in S is simply $x - vt$. This is illustrated in Figure 10.

However, the x' axis of S' has been length contracted by a factor of γ (think about the axis as a long physical ruler), so the distance from the event to the origin of S' will be $\gamma(x - vt)$ when measured in S', giving us the transformation rule

$$x' = \gamma(x - vt). \tag{33}$$

We can repeat the analysis with an event that occurred at x' and t' as measured in S' and ask what the coordinates of the event are in S. Since this is exactly the same situation as above, except now S is moving at $-v$ and with respect to S', the answer must simply be

$$x = \gamma(x' + vt'). \tag{34}$$

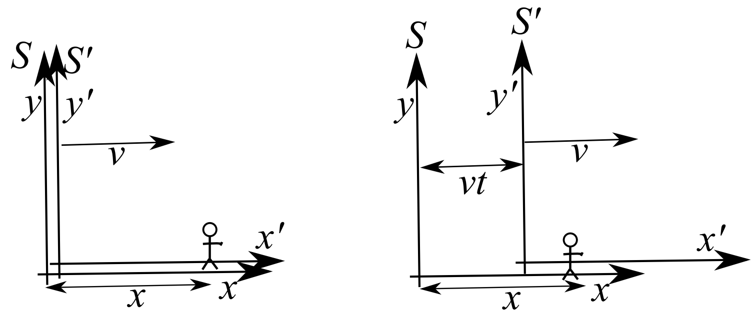


Figure 10: Two frames, S and S', in the standard configuration at $t = t' = 0$ (left) and a time t later as measured in S (right). A man waits at coordinate x in S and makes a mark on the x' axis at time t. Where is the mark on the x' axis?

8.2 Transformation of t

Eliminating x' between these two equations gives us

$$x = \gamma(\gamma(x - vt) + vt'), \tag{35}$$

which can be rearranged (*Tricky! Do this as an exercise*) to give

$$t' = \gamma(t - vx/c^2) \tag{36}$$

and similarly

$$t = \gamma(t' + vx'/c^2). \tag{37}$$

8.3 Transformations of y and z

The transformations for y and z are unchanged:

$$y' = y \tag{38}$$
$$z' = z. \tag{39}$$

This can be understood using a simple symmetry argument. If y' was not equal to y it would have to be either larger or smaller than y, and the sign of the correction would have to depend on the sign of v (so that transforming and then inverse transforming y returns y). However, the apparent sign of v can be changed by reversing the labeling on the x axis so that x increases moving left not right. Since relabeling x should not change the y coordinates, there can be no such correction. Exactly the same arguments can be applied to the z transformation.

8.4 Use of Lorentz Transformations

Rather than write down separate space and time coordinates for objects, we must always write down the space and time coordinates of events, since when the event is viewed in a different frame the new time coordinate will depend on the old space coordinate and vice versa. However, rather than specify the coordinates of an event with respect to the origin of a frame, it is often useful to specify the difference in coordinates between two events. If event one happens at (x_1, y_1, z_1, t_1) and event 2 at (x_2, y_2, z_2, t_2) then we are generally interested in the separations between the two events, $\Delta x = x_2 - x_1$, $\Delta t = t_2 - t_1$ etc. These differences transform in exactly the same way as the coordinates,[2]

[2] Which is obvious since really the coordinates are just differences between the event we are interested in and a reference event that happened at $t = x = y = z = 0$

$$\Delta x' = \gamma(\Delta x - v\Delta t) \tag{40}$$
$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2). \tag{41}$$

Exercise: Prove the time dilation and length contraction results using the Lorentz transformations. For length contraction, remember we measure length by making simultaneous measurements of the position of the two ends of our object.

9 Loss of simultaneity

An immediate consequence of the Lorentz transformations is that observers moving at different speeds do not just disagree on the time interval between events, they can also disagree about the order of events. If two events are simultaneous in S but separated in space by a distance Δx along the x axis then what happens, then the separation in time of the two events in S' is simply (applying the Lorentz transformation for Δt)

$$\Delta t' = -\gamma v\Delta x/c^2, \tag{42}$$

so events that are simultaneous is S are not simultaneous in S'. This emphasizes the importance of giving the full space-time coordinates of events, since even the order of events need not be agreed upon by different observers.

We can illustrate the loss of simultaneity with a simple example: two clocks are placed a distance Δx apart and at rest in a frame S, and a light source is placed half way between them. The light source emits a short pulse and each clock is activated when the light from the source hits it. In the frame S these events are simultaneous, so the clocks tick in time, and start ticking a time $\Delta x/(2c)$ after the pulse is turned on. However, in a frame S' moving with speed v with respect to S the light pules still moves at speed c and is still emitted from a point half way between the clocks but, because one clock is moving towards this point and one is moving away from it, the light reaches the two clocks at different times, and hence the clocks do not tick together.

Exercise: Show from first principles that the time $\Delta t'$ between the light hitting the two clocks in S' agrees with the general Lorentz transformation result stated above.

10 Addition of velocities

If, in a frame S a particle is moving with a speed u along the x axis, how fast does the particle appear to move in S'? This question can be answered straightforwardly using the difference forms of the Lorentz transformations, since if we measure the position of the particle at two different times

$$u = \frac{\Delta x}{\Delta t}. \tag{43}$$

The speed observed in S' will simply be

$$u' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - v\Delta t)}{\gamma(\Delta t - v\Delta x/c^2)}. \tag{44}$$

We can simplify this considerably by canceling γ and dividing through by Δt to get

$$u' = \frac{u - v}{1 - vu/c^2}. \tag{45}$$

This transformation law for velocities has two interesting properties. The first is that if $u = c$ then $u' = c$ whatever the value of v. This means that a particle that moves at the speed of light in one frame moves at the speed of light in every frame. The second is that if $u < c$ then it is impossible to get $u' > c$ if $v < c$, so, unlike in Galilean relativity, it is impossible to add up lots of velocities which are smaller than the speed of light to get one that is larger than the speed of light.

11 Space-Time

The Lorentz transformations mix up the time and space coordinates, so that different observers cannot agree on the distance between events, the time between events or even the order of events. It is natural to wonder whether there is anything all observers can agree on. We can make a very informative analogy with rotations of coordinate systems. If we consider two points with coordinates (x_1, y_1, z_1) and $(x_2, y_2 z_2)$, we can construct a vector that points between these two positions, $\mathbf{\Delta r} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$. If we then rotate the coordinate system while keeping the points and hence the vector fixed then the values of x_1, y_1 and z_1 etc. will all change. However, the length of the vector given by $|\mathbf{\Delta r}|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ will not have changed. We could call this unchanging quantity the space-interval between the two points, although in fact it is just the distance between them. The distance between two events does change if a Lorentz transformation is used rather than a rotation, but there in an equivalent unchanging quantity called the space-time interval between two events, which is defined as

$$\Delta s^2 = c^2 \Delta t^2 - |\mathbf{\Delta r}|^2, \tag{46}$$

where Δt is the time between the events and $|\mathbf{\Delta r}|$ is the distance between the two events. This looks much like normal Pythagoras, but the sign is different. If two observers in different frames observe the same two events, they may well disagree on the separation of the two events in time and the spatial distance between them, but they will agree on the space-time interval between the two events Δs^2. This suggests that we should think of events as points in a four dimensional space-time, in which the distances between points are not given by the usual Euclidean Pythagoras result but by the space-time interval.

Important Exercise: Use Lorentz transformations to show that two observers in different frames agree on the space-time interval between two events, even if the disagree about everything else.

12 Causality and the Universe's Speed Limit

Since observers can no longer agree on the order of events, we might justifiably be concerned about causality. If one observer sees one event cause another, how can a second observer explain this if he sees the two events occur in the other order? Surely an event cannot cause an event that happens before it. If the time interval between two events in a frame S is Δt (which, without loss of generality, we take to be positive), and they are spatially separated by a distance Δx along the x axis then we know that the time interval in another frame S' is given by

$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2). \tag{47}$$

The observers will disagree on the order of the two events if $\Delta t'$ has the opposite sign to Δt. This is only possible if

$$v\Delta x/c^2 > \Delta t. \tag{48}$$

Assuming that our observers cannot exceed the speed of light (otherwise the γ factor would be imaginary), the largest possible value of the left had side is given by setting $v = c$. This means that an observer can be found who sees the order of the events reversed if and only if

$$\Delta x > c\Delta t. \tag{49}$$

Squaring this inequality, we see that this is equivalent to demanding that the space-time interval between the two events be negative. If the space-time interval between two events is positive then the order of the event is the same according to all observers, if it is negative then observers can disagree about the order. We can also see this by considering the form of the space-time interval,

$$\Delta s^2 = c^2\Delta t^2 - |\mathbf{\Delta r}|^2. \tag{50}$$

Since both Δt^2 and $|\mathbf{\Delta r}|^2$ are positive (they are squared numbers) if the space-time interval is positive then there can be no frame in which the events are simultaneous, while if the space-time interval is negative then there can be no frame in which the two events occur at the same place. If $\Delta t'$ changes sign at some value of v then, since it is a smooth function, it must pass through zero, which cannot happen if the space-time interval is positive.

For one event to cause another, a signal must pass from the first event to the second telling the second event to happen. If the space-time interval between the events is positive then the speed of such a signal, $|\Delta x/\Delta t|$, can be less than the speed of light, while if it is negative the signal must travel faster than the speed of light. Thankfully this is consistent with our observations about which events have a definite order. If the space-time interval is positive, all observers agree on the order of two events and a signal traveling slower than the speed of light can pass between them, so the first event can cause the second. If the space-time interval is negative then observers disagree on the order of the events, so it must be impossible for one to cause the other. If one was to cause the other, it would need to send a signal faster than the speed of light. Since we cannot tolerate a theory in which events in the future cause events in the past, we conclude that no signal can travel faster than the speed of light.

Exercise: A space-time diagram is a two dimensional plot in which one axis is the spatial x axis and the other is a time axis, so that an event is represented by one point on the diagram. On the space-time diagram shown below, which regions represent events that can be caused by the event at the origin, which represent events that can cause the event at the origin and which events can have no causal relationship with the event at the origin?